HEREDITARILY INDECOMPOSABLE BANACH ALGEBRAS OF DIAGONAL OPERATORS

SPIROS A. ARGYROS, IRENE DELIYANNI, AND ANDREAS G. TOLIAS

Abstract. We provide a characterization of the Banach spaces X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ which have the property that the dual space X^* is naturally isomorphic to the space $\mathcal{L}_{\text{diag}}(X)$ of diagonal operators with respect to $(e_n)_{n\in\mathbb{N}}$. We also construct a Hereditarily Indecomposable Banach space $\mathfrak{X}_D$ with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that $\mathfrak{X}_D^*$ is isometric to $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ with these Banach algebras being Hereditarily Indecomposable. Finally, we show that every $T\in\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is of the form $T=\lambda I+K$, where K is a compact operator.

Contents

1. Introduction

The starting point of this paper is a result connecting the dual space X^* of a space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$, with the space $\mathcal{L}_{\text{diag}}(X)$ of the diagonal operators with respect to this basis. We recall that $\mathcal{L}_{\text{diag}}(X)$ is the commutative subalgebra of $\mathcal{L}(X)$ containing all bounded linear operators T satisfying $Te_n=\lambda_n e_n$, $n\in\mathbb{N}$, for a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of scalars. As is well known, if the basis $(e_n)_{n\in\mathbb{N}}$ is unconditional, the algebra $\mathcal{L}_{\text{diag}}(X)$ is homeomorphic to the algebra $\ell_\infty(\mathbb{N})$. Our result asserts that, under some natural assumptions, the spaces X^* and $\mathcal{L}_{\text{diag}}(X)$ are isomorphic. There are classical spaces, such as the space $c(\mathbb{N})$ of all convergent sequences with the summing basis, that satisfy these conditions and thus the structure of the space of the diagonal operators acting on them is completely described. In particular, for the space $X=c(\mathbb{N})$ with the summing basis, we obtain that $\mathcal{L}_{\text{diag}}(X)$ is isometric to $\ell_1(\mathbb{N})$. Our first main result is the following.

[0] Research partially supported by ΠEBE 2007 NTUA Research Program.
2000 *Mathematics Subject Classification.* 46B28, 47L10, 46B20, 46B03.
Key words and phrases. Hereditarily Indecomposable Banach space, Banach algebra, diagonal operator, compact operator.

Theorem 1.1. Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. The following are equivalent.

(1) The map $e_n^* \mapsto e_n^* \otimes e_n$ can be extended to an isomorphism between X^* and $\mathcal{L}_{\text{diag}}(X)$.

(2) (a) The basis $(e_n)_{n\in\mathbb{N}}$ dominates the summing basis.
 (b) The norm in X^* is submultiplicative (i.e. there exists $C > 0$ such that $\| \sum_{i=1}^{n} a_i \beta_i e_i^* \| \leq C \cdot \| \sum_{i=1}^{n} a_i e_i^* \| \cdot \| \sum_{i=1}^{n} \beta_i e_i^* \|$.)

The above theorem essentially concerns conditional bases of Banach spaces. Indeed, assuming that $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis, condition (2)(a) yields that $(e_n)_{n\in\mathbb{N}}$ is equivalent to the standard basis of $\ell_1(\mathbb{N})$.

As a consequence of Theorem 1.1 we obtain the following.

Theorem 1.2. For every Banach space Z with an unconditional subsymmetric basis $(z_n)_{n\in\mathbb{N}}$ there exists a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that Z^* is isomorphic to a complemented subspace of $\mathcal{L}_{\text{diag}}(X)$.

Theorem 1.1 also yields the existence of a variety of Banach spaces X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ sharing the property that X^* is isomorphic to $\mathcal{L}_{\text{diag}}(X)$. In particular, using a slight modification of the James tree space ([15]), we obtain the following.

Theorem 1.3. There exists a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that X^* is isomorphic to $\mathcal{L}_{\text{diag}}(X)$, X^* is nonseparable and does not contain $\ell_1(\mathbb{N})$ or $c_0(\mathbb{N})$.

Let us also mention that A. Sersouri has shown in [18] that if the basis $(e_n)_{n\in\mathbb{N}}$ of the Banach space X is either boundedly complete or shrinking, then $\mathcal{L}_{\text{diag}}(X)$ coincides with the second dual of the space $\mathcal{K}_{\text{diag}}(X)$ of compact diagonal operators.

In the second part of the paper we construct a Hereditarily Indecomposable (HI) Banach space $\mathfrak{X}_D$ with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is also HI. More precisely, the following is shown.

Theorem 1.4. There exists a quasi reflexive HI Banach space $\mathfrak{X}_D$ with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ satisfying the following.

(i) The space $\mathfrak{X}_D^*$ is Hereditarily Indecomposable.
(ii) The spaces $\mathfrak{X}_D^*$ and $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ are isometric.
(iii) Every $T \in \mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is of the form $T = \lambda I + K$, where K is a compact operator.

As pointed out in [4], for every Banach space X, the space X^* is isomorphic to a complemented subspace of $\mathcal{L}(X)$ consisting of rank one operators. Therefore neither $\mathcal{L}(X)$ nor $\mathcal{K}(X)$ can be indecomposable spaces. Also, as shown in [3], there exist HI spaces having strictly singular non-compact diagonal operators, therefore one could not expect property (iii) to hold in general within the class of HI spaces with a Schauder basis. Recently, R. Haydon and the first named author ([4]) have presented a $\mathcal{L}^\infty$ HI space $\mathfrak{X}_K$ such that every $T \in \mathcal{L}(\mathfrak{X}_K)$ is of the form $T = \lambda I + K$ with K a compact operator. However, the scalar plus compact problem remains open for reflexive Banach spaces. The weaker question whether there exists a reflexive space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that every diagonal operator T is of the form $\lambda I + K$ where K is compact, is also open and its solution could

be a first step towards the solution of the general one. The systematic study of the existence of strictly singular non-compact operators on the known HI spaces (see [1], [3], [10], [12], [13]), indicates that an example of a reflexive space answering the scalar plus compact problem, in the general or the weaker form, requires new approaches of HI constructions.

The paper is organized as follows. Section 2 contains the more precise statement of Theorem 1.1 (Theorem 2.4). Its proof uses rather standard arguments. We also present the proofs of Theorems 1.2 and 1.3.

The rest of the paper is devoted to the space $\mathfrak{X}_D$. In section 3, we define its norming set D which is a subset of $c_{00}(\mathbb{N})$. The space $\mathfrak{X}_D$ is the completion of $(c_{00}(\mathbb{N}), \| \|_D)$, where $\| \|_D$ is the norm induced on $c_{00}(\mathbb{N})$ by D. For the definition of D we use as a ground set the set $G = \{\pm\chi_I : \ I \text{ finite interval of } \mathbb{N}\}$ and we apply saturation with respect to the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$. As usual, for even indices j we apply full saturation, while the operations corresponding to odd indices j are used in order to define the special functionals as in all previous HI constructions, initialized by the W.T. Gowers and B. Maurey example [14]. Once more, the present construction can be viewed as an HI extension of a ground set G, according to the approach of [8] and [7].

The novelty of the present construction arises from the need to impose a Banach algebra structure on the dual of the space. For this purpose, in each inductive step of the definition of the norming set $D = \bigcup_{n=0}^{\infty} D_n$, we close the set D_n under the pointwise products of its elements. This is necessary in order to apply Theorem 1.1 and get the isomorphism between $\mathfrak{X}_D^*$ and $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$. As shown in [8], for a ground set $G \subset c_{00}(\mathbb{N})$ such that X_G does not contain any isomorphic copy of $\ell_1(\mathbb{N})$, there exists a $D_G \subset c_{00}(\mathbb{N})$ containing G such that X_{D_G} is HI. It is worth pointing out that there are ground sets G as above, such that for any D_G containing G with D_G closed under pointwise products of its elements, the corresponding space X_{D_G} is decomposable and hence is not HI. For example, let $L \subset \mathbb{N}$ such that both $L, \mathbb{N} \setminus L$ are infinite and consider the ground set $G = \{\pm\chi_I, \ \pm\chi_{L\cap I}, \ I \text{ finite interval of } \mathbb{N}\}$. Then for every extension D_G of G with D_G being closed under pointwise products, the corresponding space X_{D_G} is decomposable. Indeed, it is easy to see that the subspace $X_L = \overline{\text{span}}\{e_n : \ n \in L\}$ is complemented in X_{D_G}.

Section 4 is devoted to the basic inequality and some of its consequences. The basic inequality is the main tool for providing upper estimates for the action of functionals of D on averages of Rapidly Increasing Sequences. Its proof in the present paper is similar to the proof of the corresponding result in [3]. In section 5, we proceed to the evaluation of the norm of averages resulting from dependent sequences with alternating signs. Our approach for this, is more complicated than in previous constructions, where this result is a direct consequence of the basic inequality and the tree structure of the special sequences. This is due to the fact that closing the set D under pointwise products of its elements, we enlarge the unconditional structure of the spaces $\mathfrak{X}_D$, $\mathfrak{X}_D^*$. Thus showing the HI property of the space $\mathfrak{X}_D$ becomes more involved and delicate.

The HI property of $\mathfrak{X}_D^*$ is proved in section 6. The isomorphism between $\mathfrak{X}_D^*$ and $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D, (e_n)_{n\in\mathbb{N}})$ is an immediate consequence of Theorem 1.1 and the definition of the norming set D.

We close this section by pointing out that it is not clear if, for every Schauder basis $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_D$, the corresponding space $\mathcal{L}_{\mathrm{diag}}(\mathfrak{X}_D,(x_n)_{n\in\mathbb{N}})$ is isomorphic to $\mathfrak{X}_D^*$ or if it is Hereditarily Indecomposable.

2. On the isomorphism between $\mathcal{L}_{\mathrm{diag}}(X)$ and X^*

In this section, we give the precise statement and the proof of the characterization of the Banach spaces X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ which have the property that the dual space X^* is naturally isomorphic the space $\mathcal{L}_{\mathrm{diag}}(X)$ of the diagonal operators with respect to $(e_n)_{n\in\mathbb{N}}$. We also state and prove Theorems 1.2 and 1.3.

We start with some preliminary notation. We denote by $c_{00}(\mathbb{N})$ the space of all eventually zero sequences of reals, by $e_1,e_2,\dots$ its standard Hamel basis, while for $x=\sum a_ie_i\in c_{00}(\mathbb{N})$ the support $\operatorname{supp} x$ of x is the set $\operatorname{supp} x=\{i\in\mathbb{N}:\ a_i\neq 0\}$. For E,F nonempty finite subsets of $\mathbb{N}$, we write $E<F$ if $\max E<\min F$, while for nonzero $x,y\in c_{00}(\mathbb{N})$ we write $x<y$ if $\operatorname{supp} x<\operatorname{supp} y$. For $x,y\in c_{00}(\mathbb{N})$, $x=\sum a_ie_i$, $y=\sum\beta_ie_i$ the pointwise product of x,y is the vector $x\cdot y=\sum a_i\beta_ie_i$. For $f\in c_{00}(\mathbb{N})$ and $E\subset\mathbb{N}$ we denote by Ef the pointwise product $\chi_E\cdot f$. For a finite set F, we denote its cardinality by $\#F$. For $K,L\subset c_{00}(\mathbb{N})$ we denote $K+L=\{f+g:\ f\in K,\ g\in L\}$ and $K\cdot L=\{f\cdot g:\ f\in K,\ g\in L\}$.

Notation 2.1. Let $(X,\|\cdot\|)$ be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. We denote by $\mathcal{L}_{\mathrm{diag}}(X,(e_n)_{n\in\mathbb{N}})$ the algebra of all bounded linear diagonal operators of X with respect to the basis $(e_n)_{n\in\mathbb{N}}$, i.e.

$$\mathcal{L}_{\mathrm{diag}}(X,(e_n)_{n\in\mathbb{N}})=\{T\in\mathcal{L}(X):\ \exists(\lambda_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}} \text{ such that } Te_n=\lambda_ne_n,\ \forall n\}.$$

When the basis is a priori fixed we use the notation $\mathcal{L}_{\mathrm{diag}}(X)$.

For every n, we denote by $\overline{\mathbf{e}}_n$ the rank one operator $\overline{\mathbf{e}}_n=e_n^*\otimes e_n:X\to X$, i.e. the diagonal operator defined by the rule $\overline{\mathbf{e}}_n(\sum\limits_{i=1}^{\infty}\mu_ie_i)=\mu_ne_n$.

Remark 2.2. In the case the basis $(e_n)_{n\in\mathbb{N}}$ of the space X is unconditional, the algebra $\mathcal{L}_{\mathrm{diag}}(X)$ is isomorphic to $\ell_\infty(\mathbb{N})$.

Definition 2.3. Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. We say that $(e_n)_{n\in\mathbb{N}}$ dominates the summing basis with constant C_1, if for every finite sequence of scalars $(\mu_i)_{i=1}^n$ it holds that $C_1\cdot|\sum\limits_{i=1}^n\mu_i|\leq\|\sum\limits_{i=1}^n\mu_ie_i\|$.

Theorem 2.4. Let $(X,\|\cdot\|)$ be a Banach space with a normalized monotone Schauder basis $(e_n)_{n\in\mathbb{N}}$. Let also $C_1,C_2>0$. The statements (1), (2), (3) are equivalent:

(1) The operator $\Phi:X^*\to\mathcal{L}_{\mathrm{diag}}(X)$ defined by the rule

$$w^*-\sum_{n=1}^{\infty}\lambda_ne_n^*\ \longmapsto\ SOT-\sum_{n=1}^{\infty}\lambda_n\overline{\mathbf{e}}_n$$

is well defined, onto and $C_1\cdot\|\sum\limits_{n=1}^{\infty}\lambda_ne_n^*\|\leq\|\sum\limits_{n=1}^{\infty}\lambda_n\overline{\mathbf{e}}_n\|\leq C_2\cdot\|\sum\limits_{n=1}^{\infty}\lambda_ne_n^*\|$ for every $x^*=w^*-\sum\limits_{n=1}^{\infty}\lambda_ne_n^*\in X^*$.

(2) (a) The Schauder basis $(e_n)_{n\in\mathbb{N}}$ dominates the summing basis with constant C_1.

(b) The norm of X^* is submultiplicative with constant C_2, i.e.

$$\|\sum_{i=1}^{n} a_i\beta_i e_i^*\| \leq C_2 \cdot \|\sum_{i=1}^{n} a_i e_i^*\| \cdot \|\sum_{i=1}^{n} \beta_i e_i^*\|$$

for every $n \in \mathbb{N}$ and every choice of scalars $a_1, \beta_1, a_2, \beta_2, \dots, a_n, \beta_n \in \mathbb{R}$.
(3) There exists a 1-norming set K of X, contained in the linear span of $(e_n^*)_{n\in\mathbb{N}}$, such that

(a) $\pm C_1 \cdot \sum\limits_{i=1}^{n} e_i^* \in K$ for all n.

(b) $K \cdot K \subset C_2 \cdot B_{X^*}$.

Proof. We first show that (2) $\Longrightarrow$ (3). Suppose that (2) holds and set

$$K = \Big\{ \sum_{i=1}^{n} a_i e_i^* : \ \|\sum_{i=1}^{n} a_i e_i^*\| \leq 1, \ n \in \mathbb{N}\Big\}.$$

It is clear that (3)(b) is satisfied, as a consequence of (2)(b). Let us verify (3)(a). For every $x = \sum\limits_{i=1}^{\infty} \mu_i e_i \in B_X$, using condition (2)(a) and the monotonicity of the Schauder basis $(e_i)_{i\in\mathbb{N}}$, we get that

$$|(\sum_{i=1}^{n} e_i^*)(x)| = |\sum_{i=1}^{n} \mu_i| \leq \frac{1}{C_1} \cdot \|\sum_{i=1}^{n} \mu_i e_i\| \leq \frac{1}{C_1} \cdot \|\sum_{i=1}^{\infty} \mu_i e_i\| \leq \frac{1}{C_1}.$$

This implies that $\|\sum\limits_{i=1}^{n} e_i^*\| \leq \frac{1}{C_1}$, hence $\pm C_1 \cdot \sum\limits_{i=1}^{n} e_i^* \in K$.

Let us show the inverse implication (3) $\Longrightarrow$ (2). Suppose that (3) holds and let $n \in \mathbb{N}$, $\mu_1, \dots, \mu_n \in \mathbb{R}$. Since $\pm C_1 \sum\limits_{i=1}^{n} e_i^* \in K$ the action of these functionals on the vector $\sum\limits_{i=1}^{n} \mu_i e_i$ implies that $\|\sum\limits_{i=1}^{n} \mu_i e_i\| \geq C_1 \cdot |\sum\limits_{i=1}^{n} \mu_i|$, thus (2)(a) is satisfied. From condition (3)(b) we get that $\mathrm{conv}(K) \cdot \mathrm{conv}(K) \subset C_2 \cdot B_{X^*}$ and hence $\overline{\mathrm{conv}(K)}^{w^*} \cdot \overline{\mathrm{conv}(K)}^{w^*} \subset C_2 \cdot B_{X^*}$. Since K is a 1-norming set of the space X, this means that $B_{X^*} \cdot B_{X^*} \subset C_2 \cdot B_{X^*}$, which yields (2)(b).

Next we show the implication (1) $\Longrightarrow$ (2). Suppose that (1) is true. We shall first show (2)(a). We observe that for every $n \in \mathbb{N}$ and $x = \sum\limits_{i=1}^{\infty} \mu_i e_i \in X$ we have that $\|(\sum\limits_{i=1}^{n} \overline{\mathbf{e}}_i)(x)\| = \|\sum\limits_{i=1}^{n} \mu_i e_i\| \leq \|x\|$, as a consequence of the monotonicity of the basis. Thus $\|\sum\limits_{i=1}^{n} \overline{\mathbf{e}}_i\| \leq 1$, which implies that $C_1\|\sum\limits_{i=1}^{n} e_i^*\| \leq 1$. Therefore for any $\mu_1, \dots, \mu_n \in \mathbb{R}$ we get that

$$C_1|\sum_{i=1}^{n} \mu_i| = C_1|(\sum_{i=1}^{n} e_i^*)(\sum_{i=1}^{n} \mu_i e_i)| \leq C_1\|\sum_{i=1}^{n} e_i^*\| \cdot \|\sum_{i=1}^{n} \mu_i e_i\| \leq \|\sum_{i=1}^{n} \mu_i e_i\|$$

and we have shown (2)(a).

Let now $n\in\mathbb{N}$ and $a_1,\beta_1,a_2,\beta_2,\dots,a_n,\beta_n\in\mathbb{R}$. We choose $x=\sum\limits_{i=1}^{\infty}\mu_ie_i\in B_X$, such that $\|\sum\limits_{i=1}^{n}a_i\beta_ie_i^*\|=(\sum\limits_{i=1}^{n}a_i\beta_ie_i^*)(x)=\sum\limits_{i=1}^{n}a_i\beta_i\mu_i$. Then

$$\begin{aligned}\|\sum_{i=1}^{n}a_i\beta_ie_i^*\| &= (\sum_{i=1}^{n}a_ie_i^*)(\sum_{i=1}^{n}\beta_i\mu_ie_i)\le\|\sum_{i=1}^{n}a_ie_i^*\|\cdot\|\sum_{i=1}^{n}\beta_i\mu_ie_i\|\\ &= \|\sum_{i=1}^{n}a_ie_i^*\|\cdot\|(\sum_{i=1}^{n}\beta_i\overline{\mathbf{e}}_i)(\sum_{i=1}^{n}\mu_ie_i)\|\\ &\le \|\sum_{i=1}^{n}a_ie_i^*\|\cdot\|\sum_{i=1}^{n}\beta_i\overline{\mathbf{e}}_i\|\cdot\|\sum_{i=1}^{n}\mu_ie_i\|\\ &\le \|\sum_{i=1}^{n}a_ie_i^*\|\cdot C_2\|\sum_{i=1}^{n}\beta_ie_i^*\|\cdot\|x\|\\ &\le C_2\cdot\|\sum_{i=1}^{n}a_ie_i^*\|\cdot\|\sum_{i=1}^{n}\beta_ie_i^*\|\end{aligned}$$

which completes the proof of (2)(b).

Finally we prove that (2) $\Longrightarrow$ (1). Suppose that (2) holds. We start with the following claim.

Claim. The series $\sum\limits_{n=1}^{\infty}\lambda_ne_n^*$ is w^* convergent in X^* if and only if the series $\sum\limits_{n=1}^{\infty}\lambda_n\overline{\mathbf{e}}_n$ converges in the strong operator topology in $\mathcal{L}_{\mathrm{diag}}(X)$.

Proof of the claim. Suppose first that the series $\sum\limits_{n=1}^{\infty}\lambda_ne_n^*$ is w^* convergent. We choose $M>0$ such that $\|\sum\limits_{i=n}^{m}\lambda_ie_i^*\|\le M$ for all $n\le m$. We consider an arbitrary $x\in X$, $x=\sum\limits_{i=1}^{\infty}\mu_ie_i$, and we shall show that the sequence $\big((\sum\limits_{i=1}^{n}\lambda_i\overline{\mathbf{e}}_i)(x)\big)_{n\in\mathbb{N}}$ $\big($i.e. the sequence $(\sum\limits_{i=1}^{n}\lambda_i\mu_ie_i)_{n\in\mathbb{N}}\big)$ is a Cauchy sequence in X. Let $\varepsilon>0$. We choose $n_0\in\mathbb{N}$ such that $\|\sum\limits_{i=n_0}^{\infty}\mu_ie_i\|<\frac{\varepsilon}{MC_2}$. Let now any $m\ge n\ge n_0$. We select $z^*\in B_{X^*}$, with $z^*=\sum\limits_{i=1}^{\infty}\nu_ie_i^*$ as a w^* series, such that $\|\sum\limits_{i=n}^{m}\lambda_i\mu_ie_i\|=z^*(\sum\limits_{i=n}^{m}\lambda_i\mu_ie_i)=\sum\limits_{i=n}^{m}\lambda_i\mu_i\nu_i$. We set $f=\sum\limits_{i=n}^{m}\lambda_i\nu_ie_i^*$. From our assumption (2)(b) we get that $\|f\|\le C_2\cdot\|\sum\limits_{i=n}^{m}\lambda_ie_i^*\|\cdot\|\sum\limits_{i=n}^{m}\nu_ie_i^*\|\le C_2M\|z^*\|\le C_2M$. Thus $\|(\sum\limits_{i=n}^{m}\lambda_i\overline{\mathbf{e}}_i)(x)\|=\|\sum\limits_{i=n}^{m}\lambda_i\mu_ie_i\|=\sum\limits_{i=n}^{m}\lambda_i\mu_i\nu_i=f(\sum\limits_{i=n}^{m}\mu_ie_i)\le\|f\|\cdot\|\sum\limits_{i=n}^{m}\mu_ie_i\|<C_2M\frac{\varepsilon}{MC_2}=\varepsilon$.

Conversely, suppose that the series $\sum\limits_{n=1}^{\infty}\lambda_n\overline{\mathbf{e}}_n$ converges in the strong operator topology; we shall prove that the series $\sum\limits_{n=1}^{\infty}\lambda_ne_n^*$ is w^* convergent. Let $x\in$

X, $x = \sum_{i=1}^{\infty} \mu_i e_i$. We shall show that the sequence $\big((\sum_{i=1}^{n} \lambda_i e_i^*)(x)\big)_{n\in\mathbb{N}}$ (i.e. the sequence $(\sum_{i=1}^{n} \lambda_i \mu_i)_{n\in\mathbb{N}}$) is a Cauchy sequence in $\mathbb{R}$. Let $\varepsilon > 0$. From our assumption that the series $\sum_{n=1}^{\infty} \lambda_n \overline{\mathbf{e}}_n$ is SOT-convergent it follows that the sequence $\big((\sum_{i=1}^{n} \lambda_i \overline{\mathbf{e}}_i)(x)\big)_{n\in\mathbb{N}}$ converges in norm. Thus we may choose n_0 such that $\|(\sum_{i=n}^{m} \lambda_i \overline{\mathbf{e}}_i)(x)\| < C_1\varepsilon$ for every $m \geq n \geq n_0$. From assumption (2)(a) we get that $|\sum_{i=n}^{m} \lambda_i\mu_i| \leq \frac{1}{C_1}\|\sum_{i=n}^{m} \lambda_i \mu_i e_i\| = \frac{1}{C_1}\|(\sum_{i=n}^{m} \lambda_i \overline{\mathbf{e}}_i)(x)\| < \frac{1}{C_1} C_1 \varepsilon = \varepsilon$. This completes the proof of the claim. $\square$

From the first part of the claim it follows that the operator $\Phi : X^* \to \mathcal{L}_{\text{diag}}(X)$ is well defined. Taking into account that for $T \in \mathcal{L}_{\text{diag}}(X)$, if $T(e_n) = \lambda_n e_n$, $n = 1, 2, \ldots$ then $T = SOT - \sum_{n=1}^{\infty} \lambda_n \overline{\mathbf{e}}_n$, the second part of the claim entails that the operator Φ is onto $\mathcal{L}_{\text{diag}}(X)$. We shall show that $C_1 \cdot \|\sum_{i=1}^{\infty} \lambda_i e_i^*\| \leq \|\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i\| \leq C_2 \cdot \|\sum_{i=1}^{\infty} \lambda_i e_i^*\|$ for every $x^* = w^* - \sum_{i=1}^{\infty} \lambda_i e_i^* \in X^*$. From now on we fix a functional $x^* = w^* - \sum_{i=1}^{\infty} \lambda_i e_i^* \in X^*$.

From (2)(a) we get that

$$\begin{aligned}
\|\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i\| &= \sup\{\|(\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i)(\sum_{i=1}^{\infty} \mu_i e_i)\| : \sum_{i=1}^{\infty} \mu_i e_i \in B_X\} \\
&= \sup\{\|\sum_{i=1}^{n} \lambda_i \mu_i e_i\| : \sum_{i=1}^{\infty} \mu_i e_i \in B_X,\ n \in \mathbb{N}\} \\
&\geq \sup\{C_1 \cdot |\sum_{i=1}^{n} \lambda_i \mu_i| : \sum_{i=1}^{\infty} \mu_i e_i \in B_X,\ n \in \mathbb{N}\} \\
&= C_1 \cdot \sup\{|(\sum_{i=1}^{\infty} \lambda_i e_i^*)(\sum_{i=1}^{n} \mu_i e_i)| : \sum_{i=1}^{\infty} \mu_i e_i \in B_X,\ n \in \mathbb{N}\} \\
&= C_1 \cdot \|\sum_{i=1}^{\infty} \lambda_i e_i^*\|.
\end{aligned}$$

We finally show that $\|\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i\| \leq C_2 \cdot \|\sum_{i=1}^{\infty} \lambda_i e_i^*\|$. Let $\varepsilon > 0$. We select a finitely supported vector $x \in B_X$, $x = \sum_{i=1}^{n} \mu_i e_i$, such that $\|\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i\| \leq (1 + \varepsilon)\|(\sum_{i=1}^{\infty} \lambda_i \overline{\mathbf{e}}_i)(x)\| = (1+\varepsilon)\|\sum_{i=1}^{n} \lambda_i \mu_i e_i\|$. We choose $z^* = w^* - \sum_{i=1}^{\infty} \nu_i e_i^* \in B_{X^*}$ such that $\|\sum_{i=1}^{n} \lambda_i \mu_i e_i\| = z^*(\sum_{i=1}^{n} \lambda_i \mu_i e_i) = \sum_{i=1}^{n} \lambda_i \mu_i \nu_i$. We set $f = \sum_{i=1}^{n} \lambda_i \nu_i e_i^*$. From our assumption (2)(b) we get that $\|f\| \leq C_2 \cdot \|\sum_{i=1}^{n} \lambda_i e_i^*\| \cdot \|\sum_{i=1}^{n} \nu_i e_i^*\| \leq C_2 \cdot \|\sum_{i=1}^{\infty} \lambda_i e_i^*\|$.

Therefore

$$\begin{aligned}\|\sum_{i=1}^{\infty}\lambda_i\overline{\mathbf{e}}_i\| &\leq (1+\varepsilon)\cdot\|\sum_{i=1}^{n}\lambda_i\mu_i e_i\| = (1+\varepsilon)\cdot\sum_{i=1}^{n}\lambda_i\mu_i\nu_i\\ &= (1+\varepsilon)\cdot f(x)\leq(1+\varepsilon)\cdot\|f\|\cdot\|x\|\leq(1+\varepsilon)\cdot C_2\cdot\|\sum_{i=1}^{\infty}\lambda_i e_i^*\|.\end{aligned}$$

Since this happens for every $\varepsilon>0$ we conclude that $\|\sum_{i=1}^{\infty}\lambda_i\overline{\mathbf{e}}_i\|\leq C_2\cdot\|\sum_{i=1}^{\infty}\lambda_i e_i^*\|$ and this finishes the proof of the theorem. $\square$

Remark 2.5. Usually a $K\subset c_{00}(\mathbb{N})$ is considered and the space X is defined as the completion of the normed space $(c_{00}(\mathbb{N}),\|\ \|_K)$. If for such a K it holds that $K\cdot K\subset K+\cdots+K$ (m summands) then condition (3)(b) is satisfied for $C_2=m$.

Remark 2.6. One can easily prove that under the conditions of Theorem 2.4, for every choice of scalars $(a_i)_{i\in\mathbb{N}}$, the series $\sum_{i=1}^{\infty}a_i e_i^*$ converges in norm in X^* if and only if the series $\sum_{i=1}^{\infty}a_i\overline{\mathbf{e}}_i$ converges in norm in $\mathcal{L}_{\mathrm{diag}}(X)$. Since the latter means that the operator $\sum_{i=1}^{\infty}a_i\overline{\mathbf{e}}_i$ is compact, we get that under the conditions of Theorem 2.4, the space $\mathcal{K}_{\mathrm{diag}}(X)$ of compact diagonal operators of X is naturally identified with the subspace of X^* norm generated by the biorthogonal functionals $(e_n^*)_{n\in\mathbb{N}}$.

Example 2.7. The summing basis $(s_n)_{n\in\mathbb{N}}$ of the Banach space $c(\mathbb{N})$ of all convergent sequences is monotone while the set $K=\{\pm\sum_{i=1}^{n}s_i^*:\ n\in\mathbb{N}\}$ is a norming set of the space $c(\mathbb{N})$ satisfying conditions (3)(a), (3)(b) of Theorem 2.4 with constants $C_1=1$ and $C_2=1$. Therefore, Theorem 2.4 implies that the space of all diagonal operators of the space $c(\mathbb{N})$ with respect to the summing basis, is isometric to $c(\mathbb{N})^*$, which is isometric to $\ell_1(\mathbb{N})$.

Remark 2.8. It follows readily that if the space X has an unconditional basis $(e_n)_{n\in\mathbb{N}}$, then, denoting by $(e_n^*)_{n\in\mathbb{N}}$ the corresponding biorthogonal functionals, the following holds. For every $f=w^*-\sum_{n=1}^{\infty}a_n e_n^*$, $g=w^*-\sum_{n=1}^{\infty}\beta_n e_n^*$ in X^* we have that

$$\|w^*-\sum_{n=1}^{\infty}a_n\beta_n e_n^*\|\leq C\cdot\|w^*-\sum_{n=1}^{\infty}a_n e_n^*\|\cdot\|w^*-\sum_{n=1}^{\infty}\beta_n e_n^*\|$$

where C is a constant which depends on the unconditional basis constant of $(e_n)_{n\in\mathbb{N}}$. Therefore the dual of a space with an unconditional basis is a Banach algebra.

On the other hand, it is clear that if the space X has an unconditional basis $(e_n)_{n\in\mathbb{N}}$ and it satisfies condition (2)(a) of Theorem 2.4, then X is isomorphic to $\ell_1(\mathbb{N})$. Thus, although for every space X with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ it holds that $\mathcal{L}_{\mathrm{diag}}(X,(e_n)_{n\in\mathbb{N}})$ is isomorphic to $\ell_\infty(\mathbb{N})$, the only space for which this fact follows as a consequence of Theorem 2.4, is $\ell_1(\mathbb{N})$.

Theorem 2.9. Let Z be a Banach space with an unconditional subsymmetric Schauder basis. Then there exists a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that $\mathcal{L}_{\mathrm{diag}}(X,(e_n)_{n\in\mathbb{N}})$ has a complemented subspace isomorphic to Z^*.

Proof. Passing to an equivalent norm, we may assume that the space Z has a normalized, bimonotone basis $(z_n)_{n\in\mathbb{N}}$ which is 1-unconditional and subsymmetric. This means that, given $z=\sum_{i=1}^{d}\lambda_i z_i$, for any increasing sequence $(k_i)_{i=1}^d$ we have that $\|\sum_{i=1}^{d}\lambda_i z_{k_i}\|=\|z\|$ and that for every choice of scalars $(\mu_i)_{i=1}^d$ with $|\mu_i|\le|\lambda_i|$ we have that $\|\sum_{i=1}^{d}\mu_i z_i\|\le\|z\|$. The same properties remain valid for the sequence of biorthogonal functionals $(z_n^*)_{n\in\mathbb{N}}$.

The space X is defined to be the Jamesification J_Z of the space Z (see also [11]). Namely, setting

$$\begin{aligned} K &= \Big\{\sum_{i=1}^{d} a_i\chi_{I_i} :\ I_1<I_2<\cdots<I_d \text{ are finite intervals,} \\ &\qquad\qquad \|\sum_{i=1}^{d} a_i z_i^*\|_{Z^*}\le 1,\ d\in\mathbb{N}\Big\} \end{aligned}$$

the space $X=J_Z$ is the completion of $(c_{00}(\mathbb{N}),\|\ \|_K)$. The Hamel basis $(e_n)_{n\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ becomes a normalized bimonotome Schauder basis for X, while the norming set K of X obviously satisfies condition (3)(a) of Theorem 2.4 with constant $C_1=1$.

We next show that $K\cdot K\subset K+K$. Fix $f=\sum_{i=1}^{d}a_i\chi_{I_i}$, $g=\sum_{j=1}^{d'}\beta_j\chi_{E_j}$ in K. Without loss of generality we may assume that each I_i intersects some E_j and and each E_j intersects some I_i. For each $j=1,\dots,d'$, let i_j be the minimum i for which $I_i\cap E_j\neq\emptyset$. Using that $\|\sum_{j=1}^{d'}\beta_j z_j^*\|\le 1$, $|a_i|\le 1$ for each i, the 1-unconditionality and the subsymmetricity of the basis $(z_n)_{n\in\mathbb{N}}$, we get that the functional $h_1=\sum_{j=1}^{d}a_{i_j}\beta_j\chi_{I_{i_j}\cap E_j}$ belongs to the set K. Observe that for each i, there exists at most one j such that $I_i\cap E_j\neq\emptyset$ and $i\neq i_j$. Let A be the set of all i for which such a j exists and denote this j by j_i. Using that $\|\sum_{i=1}^{d}a_i z_i^*\|\le 1$, $|b_j|\le 1$ for each j, the 1-unconditionality and the subsymmetricity of the basis $(z_n)_{n\in\mathbb{N}}$, we derive that the functional $h_2=\sum_{i\in A}a_i b_{j_i}\chi_{I_i\cap E_{j_i}}$ also belongs to the set K. Hence the functional $f\cdot g=h_1+h_2$ belongs to $K+K$, therefore (see Remark 2.5) condition (3)(b) of Theorem 2.4 is satisfied with constant $C_2=2$.

Theorem 2.4 entails that the space of diagonal operators $\mathcal{L}_{\mathrm{diag}}(X,(e_n)_{n\in\mathbb{N}})$ is isomorphic to X^*. It remains to show that the dual space Z^* is isomorphic to a complemented subspace of X^*.

We define $w_n=e_{2n-1}-e_{2n}$ for $n=1,2,\ldots$.

Claim. The basic sequence $(w_n)_{n\in\mathbb{N}}$ of X is 2-equivalent to the basis $(z_n)_{n\in\mathbb{N}}$ of Z, i.e. for every sequence of scalars $(\lambda_k)_{k=1}^d$

$$\|\sum_{k=1}^{d}\lambda_k z_k\|_Z\le\|\sum_{k=1}^{d}\lambda_k w_k\|_X\le 2\|\sum_{k=1}^{d}\lambda_k z_k\|_Z \tag{1}$$

Proof of the claim. Let $z = \sum_{k=1}^{d} \lambda_k z_k$ be a finitely supported vector in Z. We choose a functional $z^* = \sum_{k=1}^{d} a_k z_k^* \in B_{Z^*}$ such that $z^*(z) = \|z\|_Z$. Then the functional $f = \sum_{k=1}^{d} a_k e_{2k-1}^*$ belongs to K and thus

$$\|\sum_{k=1}^{d} \lambda_k w_k\|_X \geq f(\sum_{k=1}^{d} \lambda_k w_k) = \sum_{k=1}^{d} a_k \lambda_k = z^*(z) = \|\sum_{k=1}^{d} \lambda_k z_k\|_Z.$$

Next we prove the inequality in the right side of (1). Let $f = \sum_{i=1}^{d'} a_i \chi_{I_i}$ be an arbitrary functional in K. Observe that if $\min I_i$ is odd and $\max I_i$ is even then $\chi_{I_i}(w_k) = 0$ for all k. We set

$$\begin{aligned} A_0 &= \{i : \min I_i \text{ is even and } \max I_i \text{ is odd}\} \\ A_1 &= \{i : \min I_i \text{ is even and } \max I_i \text{ is even}\} \\ A_2 &= \{i : \min I_i \text{ is odd and } \max I_i \text{ is odd}\} \end{aligned}$$

For $i \in A_0 \cup A_1$ let $\min I_i = 2p_i$ and for $i \in A_0 \cup A_2$ let $\max I_i = 2q_i - 1$. It follows that

$$\begin{aligned} f(\sum_{k=1}^{d} \lambda_k w_k) &= \sum_{i \in A_0} a_i(-\lambda_{p_i} + \lambda_{q_i}) + \sum_{i \in A_1} a_i(-\lambda_{p_i}) + \sum_{i \in A_2} a_i \lambda_{q_i} \\ &= \sum_{i \in A_0 \cup A_1} (-a_i)\lambda_{p_i} + \sum_{i \in A_0 \cup A_2} a_i \lambda_{q_i} \\ &= (\sum_{i \in A_0 \cup A_1} (-a_i) z_{p_i}^*)(\sum_{k=1}^{d} \lambda_k z_k) + (\sum_{i \in A_0 \cup A_2} a_i z_{q_i}^*)(\sum_{k=1}^{d} \lambda_k z_k) \\ &\leq 2\|\sum_{k=1}^{d} \lambda_k z_k\|_Z \end{aligned}$$

where we have used the 1-unconditionality and the subsymmetricity of the basis and the fact that $\|\sum_{i=1}^{d} a_i z_i^*\| \leq 1$. Thus it follows that $\|\sum_{k=1}^{d} \lambda_k w_k\|_X \leq 2\|\sum_{k=1}^{d} \lambda_k z_k\|_Z$. $\square$

It follows from the claim that the space Z is isomorphic to the subspace $W = \overline{\text{span}}\{w_n : n \in \mathbb{N}\}$ of X. We claim that W is a complemented subspace of X. Indeed, let $P : X \to W$ with $P(\sum_{n=1}^{\infty} \lambda_n e_n) = \sum_{n=1}^{\infty} \lambda_{2n-1} w_n$. Then, for any choice of scalars $(\lambda_n)_{n=1}^{2d}$ we have that

$$\|\sum_{n=1}^{d} \lambda_{2n-1} w_n\|_X \leq 2\|\sum_{n=1}^{d} \lambda_{2n-1} z_n\|_Z \leq 2\|\sum_{n=1}^{2d} \lambda_n e_n\|_X.$$

Thus $\|P\| \leq 2$, while, since obviously $P(w_n) = w_n$ for all n, P is a projection onto W.

Therefore Z^*, being isomorphic to W^*, is isomorphic to a complemented subspace of X^*. Since $\mathcal{L}_{\text{diag}}(X, (e_n)_{n \in \mathbb{N}})$ is isomorphic to X^*, we conclude that the

space of diagonal operators $\mathcal{L}_{\text{diag}}(X,(e_n)_{n\in\mathbb{N}})$ has a complemented subspace isomorphic to Z^*. $\square$

Theorem 2.10. There exists a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ such that its space of diagonal operators $\mathcal{L}_{\text{diag}}(X)$ is nonseparable and does not contain $c_0(\mathbb{N})$ or $\ell_1(\mathbb{N})$.

Proof. The space $X = JT_I$ that we define, is a variant of the classical James Tree space JT. We consider the dyadic tree $(\mathcal{D}, \preceq)$ with its standard interpretation as the set of all finite sequences of 0's and 1's with partial order defined by the relation $a \preceq \beta$ iff a is an initial segment of β. The lexicographical order of $\mathcal{D}$ is defined through the bijection $h : \mathcal{D} \to \mathbb{N}$ defined by the rule $h(\emptyset) = 1$, $h(\varepsilon_1 \dots \varepsilon_n) = 2^n + \sum\limits_{j=1}^{n} \varepsilon_j 2^{n-j}$. Thus the set $\mathbb{N}$ of the natural numbers is endowed with a partial order $\preceq$ such that $(\mathbb{N}, \preceq)$ coincides with the dyadic tree, while the natural order of $\mathbb{N}$ coincides with the lexicographical order of $\mathcal{D}$. Thus a subset A of $\mathbb{N}$ is called a segment of the dyadic tree if the set $h^{-1}(A)$ is a segment of $\mathcal{D}$. We set

$$\begin{aligned}\mathcal{S} \;=\; & \{A : \ A \text{ is a finite interval of } \mathbb{N}\} \cup \\ & \qquad \{A : \ A \text{ is a finite segment of the dyadic tree}\}.\end{aligned}$$

We observe that for $A, B \in \mathcal{S}$ we have that $A \cap B \in \mathcal{S}$.

We define the subset K of $c_{00}(\mathbb{N})$ as

$$K = \Big\{ \sum_{i=1}^{d} a_i \chi_{A_i} : \ (A_i)_{i=1}^{d} \text{ are pairwise disjoint members of } \mathcal{S}, \ \sum_{i=1}^{d} a_i^2 \leq 1, \ d \in \mathbb{N} \Big\}.$$

The space JT_I is the completion of the normed space $(c_{00}(\mathbb{N}), \|\cdot\|_K)$. The standard Hamel basis $(e_n)_{n\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ is a normalized bimonotone Schauder basis of JT_I. Since clearly $\pm\chi_{\{1,\dots,n\}} \in K$ for all n, condition (3)(a) of Theorem 2.4 is satisfied with constant $C_1 = 1$. Also, if $f = \sum\limits_{i=1}^{d} a_i \chi_{A_i}$ and $g = \sum\limits_{j=1}^{d'} \beta_j \chi_{B_j}$ belong to K, then we have that $f \cdot g = \sum\limits_{i=1}^{d} \sum\limits_{j=1}^{d'} a_i \beta_j \chi_{A_i \cap B_j}$ with $A_i \cap B_j \in \mathcal{S}$ and $\sum\limits_{i=1}^{d} \sum\limits_{j=1}^{d'} a_i^2 \beta_j^2 \leq 1$, hence $f \cdot g \in K$. Thus condition (3)(b) of Theorem 2.4 is satisfied with constant $C_2 = 1$. Therefore Theorem 2.4 entails that the algebra $\mathcal{L}_{\text{diag}}(JT_I)$, of all diagonal operators of the space JT_I with respect to the basis $(e_n)_{n\in\mathbb{N}}$, is isometric to JT_I^*. Similarly to the proof for the classical James Tree space JT (see e.g. [15], [16]), it is proved that JT_I contains no isomorphic copy of $\ell_1(\mathbb{N})$, hence JT_I^* contains no isomorphic copy of $c_0(\mathbb{N})$, while JT_I^* is nonseparable and $(JT_I)^{**}$ is isomorphic to $JT_I \oplus \ell_2(c)$. The later implies that JT_I^* does not contain $\ell_1(\mathbb{N})$. Therefore the space of diagonal operators $\mathcal{L}_{\text{diag}}(JT_I)$ is nonseparable and does not contain $c_0(\mathbb{N})$ or $\ell_1(\mathbb{N})$. $\square$

3. Definition of the space $\mathfrak{X}_D$

The content of the present section is the definition of the space $\mathfrak{X}_D$ that we shall study in the subsequent sections. The novelty of the definition of the space $\mathfrak{X}_D$ (compared to earlier HI constructions) is that, in each inductive step of the definition of its norming set D, we close under pointwise products, in order to obtain a set D satisfying condition (3)(b) of Theorem 2.4. This forces the dual

space $\mathfrak{X}_D^*$ to be a Banach algebra. We also include in the norming set D of the space $\mathfrak{X}_D$, the functionals $\pm\chi_I$ for all finite intervals I, in order to satisfy condition (3)(a) of Theorem 2.4.

Definition 3.1 (The space $\mathfrak{X}_D$)**.** We fix two sequences of integers $(m_j)_{j\in\mathbb{N}}$, $(n_j)_{j\in\mathbb{N}}$, as follows:

- $m_1 = 2$ and $m_j = m_{j-1}^5$ for $j > 1$.
- $n_1 = 4$ and $n_j > m_j^{4\log_2(m_j)+2}\cdot Q_j^{2\log_2(m_j)}$ for $j > 1$, where $Q_j = \big(n_{j-1}\cdot \log_2(m_j)\big)^{\log_2(m_j)}$

Let $\mathbb{Q}_s$ denote the set of all finite sequences $(\phi_1,\phi_2,\dots,\phi_d)$ such that $\phi_i\in c_{00}(\mathbb{N})$, $\phi_i\neq 0$ with $\phi_i(n)\in\mathbb{Q}$ for all i,n and $\phi_1<\phi_2<\cdots<\phi_d$. We fix a pair Ω_1,Ω_2 of disjoint infinite subsets of $\mathbb{N}$. From the fact that $\mathbb{Q}_s$ is countable we are able to define a Gowers-Maurey type injective coding function $\sigma:\mathbb{Q}_s\to\{2j:\ j\in\Omega_2\}$ such that $m_{\sigma(\phi_1,\phi_2,\dots,\phi_d)}>\max\{\frac{1}{|\phi_i(e_l)|}:\ l\in\operatorname{supp}\phi_i,\ i=1,\dots,d\}\cdot\max\operatorname{supp}\phi_d$.

We shall inductively define a triple (K_n,M_n,D_n) of subsets of $c_{00}(\mathbb{N})$, $n = 0,1,2,\dots$ and K_n^j, $j=0,1,2,\dots$ with $K_n=\bigcup\limits_{j=0}^{\infty}K_n^j$.

We first define $G=\{\pm\chi_I:\ I \text{ is a finite interval of } \mathbb{N}\}$. We set $K_0^0=G$, $K_0^j=\emptyset$, $j=1,2,\dots$ and $K_0=G$, $M_0=G$, $D_0=\operatorname{conv}_{\mathbb{Q}}(G)$.

Suppose that the sets $(K_n^j)_{j=0}^{\infty}$, M_n, D_n have been defined. We set $K_{n+1}^0=G$ and for $j=1,2,\dots$ we define

$$K_{n+1}^{2j}=\Big\{\frac{1}{m_{2j}}\sum_{i=1}^{d}f_i:\ f_i\in D_n,\ f_1<\cdots<f_d,\ d\le n_{2j}\Big\}\cup K_n^{2j}$$

and

$$\begin{aligned}K_{n+1}^{2j-1} \;=\; \Big\{ &\pm E\Big(\frac{1}{m_{2j-1}}\sum_{i=1}^{n_{2j-1}}f_i\Big):\ f_1\in K_n^{2j_1} \text{ for some } j_1\in\Omega_1\\ &\text{with } m_{2j_1}^{1/2}>n_{2j-1},\ f_{i+1}\in K_n^{\sigma(f_1,\dots,f_i)},\\ &f_1<f_2<\cdots<f_{n_{2j-1}},\\ &E \text{ is an interval of } \mathbb{N}\Big\}\cup K_n^{2j-1}.\end{aligned}$$

We set

$$\begin{aligned}K_{n+1} &= \bigcup_{j=0}^{\infty}K_{n+1}^j,\\ M_{n+1} &= \{f_1\cdot\ \dots\ \cdot f_d:\ f_i\in K_{n+1},\ i=1,\dots,d,\ d\in\mathbb{N}\}\\ D_{n+1} &= \operatorname{conv}_{\mathbb{Q}}(M_{n+1}).\end{aligned}$$

The inductive construction has been completed. We finally set $K^j=\bigcup\limits_{n=0}^{\infty}K_n^j$ for $j=1,2\dots$ and $K=\bigcup\limits_{n=0}^{\infty}K_n$, $M=\bigcup\limits_{n=0}^{\infty}M_n$, $D=\bigcup\limits_{n=0}^{\infty}D_n$.

The Banach space $\mathfrak{X}_D$ is the completion of the normed space $(c_{00}(\mathbb{N}),\|\cdot\|_D)$.

Remark 3.2. The norming set D of the space $\mathfrak{X}_D$ is closed under pointwise products. Indeed, since $K_n\subset K_{n+1}$ and $K_n\subset M_n\subset D_n$ for all n, in order to show the former it is enough to show that each D_n is closed under pointwise products. Let

$f, g \in D_n$. Then $f = \sum_{i=1}^{d} \lambda_i f_i$, $g = \sum_{j=1}^{d'} \mu_j g_j$ as convex combinations, with $(f_i)_{i=1}^d$ and $(g_j)_{j=1}^{d'}$ in M_n. The pointwise product $f \cdot g$ takes the form $f \cdot g = \sum_{i=1}^{d} \sum_{j=1}^{d'} \lambda_i \mu_j \, f_i \cdot g_j$ as a convex combination of the family of functionals $(f_i \cdot g_j)_{i=1,j=1}^{d,\;d'}$ with each $f_i \cdot g_j$ belonging to M_n. Therefore $f \cdot g \in D_n \subset D$.

It follows that $\|f \cdot g\| \leq \|f\| \cdot \|g\|$ for every $f, g \in \mathfrak{X}_D^*$, therefore the space $\mathfrak{X}_D^*$ is a Banach algebra.

Remark 3.3. The set D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties:

(i) $G \subset D$, i.e. the set D contains $\pm\chi_I$ for any finite interval I of $\mathbb{N}$.
(ii) D is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})_j$ operations.
(iii) For each j, the set D is closed under the $(\mathcal{A}_{n_{2j-1}}, \frac{1}{m_{2j-1}})$ operation on $2j-1$ special sequences (see Definition 3.4 below).
(iv) D is closed under the restriction of its elements to intervals of $\mathbb{N}$.
(v) D is rationally convex.
(vi) D is closed under pointwise products.

Definition 3.4. A block sequence $(f_i)_{i=1}^{n_{2j-1}}$ is said to be a $2j-1$ special sequence if $f_1 \in K^{2j_1}$ for some $j_1 \in \Omega_1$ with $m_{2j_1}^{1/2} > n_{2j-1}$ and $f_{i+1} \in K^{\sigma(f_1,\dots,f_i)}$ for $1 \leq i < n_{2j-1}$.

Remark 3.5. The sequence $(e_n)_{n\in\mathbb{N}}$ is clearly a normalized bimonotone Schauder basis of the space $\mathfrak{X}_D$. From the fact that the norming set D is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})_j$ operations, and taking into account that $\lim_j \frac{m_j}{n_j} = 0$, it also follows that the basis $(e_n)_{n\in\mathbb{N}}$ is boundedly complete. Hence the space $(\mathfrak{X}_D)_* = \overline{\text{span}}\{e_n^* : n \in \mathbb{N}\}$ is a predual of the space $\mathfrak{X}_D$. Notice also that, as a consequence of the fact that the norming set D is rationally convex, the set D is pointwise dense in the unit ball $\mathrm{B}_{\mathfrak{X}_D^*}$ of the dual space. Since the set D is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})_{j\in\mathbb{N}}$ operations, we get that the unit ball $\mathrm{B}_{\mathfrak{X}_D^*}$ shares the same property, i.e. if $j \in \mathbb{N}$, $d \leq n_{2j}$ and $f_1 < f_2 < \cdots < f_d$ with $\|f_i\| \leq 1$ then $\|\frac{1}{m_{2j}} \sum_{i=1}^{d} f_i\| \leq 1$.

Remark 3.6. Each $f \in D$ has one of the following forms:

(a) $f = \pm\chi_I$, I an interval of $\mathbb{N}$ (i.e. $f \in G$). We set $w(f) = 1$.
(b) $f = \frac{1}{m_j} \sum_{i=1}^{d} f_i$ with $f_1 < \cdots < f_d$, $d \leq n_j$, $f_i \in D$ (i.e. $f \in K^j \subset K$). In this case we set $w(f) = m_j$.
(c) The functional $f \in M$ is the pointwise product $f = f_1 \cdot \ldots \cdot f_d$ with each $f_i \in K$. In this case we define $w(f) = w(f_1) \cdot \ldots \cdot w(f_d)$.
(d) f is a rational convex combination $f = \sum_{i=1}^{d} \lambda_i f_i$ with each f_i belonging to cases (a), (b), (c).

We next show, that a functional $f \in M$ (i.e. a pointwise product), can be written as $f = \frac{1}{w(f)} \sum h_i$ with $(h_i)_i$ being a sequence of successive functionals belonging to the norming set D, of length determined by $w(f)$.

Lemma 3.7. If $(I_i)_{i=1}^d$, $(J_j)_{j=1}^{d'}$ is any pair of families of successive intervals of $\mathbb{N}$ (i.e. $I_1 < \cdots < I_d$ and $J_1 < \cdots < J_{d'}$) then the cardinality of the nonempty sets of the family $\{I_i \cap J_j,\ 1 \leq i \leq d,\ 1 \leq j \leq d'\}$ is at most $d + d' - 1$.

Proof. We proceed by induction on the sum $d+d'$. For $d+d' = 2$, i.e. if $d = d' = 1$ there is nothing to be proved. Let $k \geq 2$ and suppose that the lemma is true for $d + d' \leq k$. We prove the result for $d + d' = k + 1$. If $d = 1$ or $d' = 1$ then the result is straightforward, so we assume that $d > 1$ and $d' > 1$. Without loss of generality we may assume that $\min J_{d'} \leq \min I_d$. From our inductive assumption the family $\{I_i \cap J_j,\ 1 \leq i \leq d-1,\ 1 \leq j \leq d'\}$ has at most $(d-1) + d' - 1$ nonempty sets. Since $\min J_{d'} \leq \min I_d$ we get that $I_d \cap J_j = \emptyset$ for $j = 1, \ldots, d'-1$. Thus the only set between $J_1, \ldots, J_{d'}$ that may intersect I_d is $J_{d'}$. Therefore the nonempty sets of the family $\{I_i \cap J_j,\ 1 \leq i \leq d,\ 1 \leq j \leq d'\}$ are at most $[(d-1) + d' - 1] + 1$ i.e. at most $d + d' - 1$. $\square$

Proposition 3.8. Let $f \in M$, $f = f_1 \cdot \ldots \cdot f_r$ with $f_i \in K$, $w(f_i) = m_{j_i}$, $i = 1, \ldots, r$. Then the functional f takes the form $f = \frac{1}{w(f)} \sum_{i=1}^{d} h_i$ where $h_i \in D$, $h_1 < \cdots < h_d$ and $d \leq n_{j_1} + \cdots + n_{j_r} - (r-1)$. Moreover, if $f \in M_{n+1}$, then we may select each h_i to belong to D_n.

Proof. Let $f \in M_{n+1}$. Then $f = f_1 \cdot \ldots \cdot f_r$ with each $f_i \in K_{n+1}$. We shall prove, by induction on k, that each product $f_1 \cdot \ldots \cdot f_k$, for $1 \leq k \leq r$, takes the desired form. For $k = 1$ there is nothing to be proved. Let $k < r$ and suppose that $f_1 \cdot \ldots \cdot f_k = \frac{1}{w(f_1) \cdot \ldots \cdot w(f_k)} \sum_{l=1}^{d'} H_l$ with $H_l \in D_n$, $H_1 < \cdots < H_{d'}$ and $d' \leq n_{j_1} + \cdots + n_{j_k} - (k-1)$. Let also $f_{k+1} = \frac{1}{m_{j_{k+1}}}(f_1^{k+1} + \cdots + f_m^{k+1})$, $m \leq n_{j_{k+1}}$, with each $f_j^{k+1} \in D_n$. Applying Lemma 3.7 to the families $(\operatorname{ran} H_l)_{l=1}^{d'}$ and $(\operatorname{ran} f_j^{k+1})_{j=1}^m$ we get that $\operatorname{ran} H_l \cap \operatorname{ran} f_j^{k+1} \neq \emptyset$ for at most $n_{j_1} + \cdots + n_{j_{k+1}} - k$ pairs (l, j). Taking into account that the set D_n is closed under pointwise products (see Remark 3.2) we get that

$$f_1 \cdot \ldots \cdot f_k \cdot f_{k+1} = \frac{1}{w(f_1) \cdot \ldots \cdot w(f_k)} \frac{1}{m_{j_{k+1}}} (\sum_{l=1}^{d'} H_l)(\sum_{j=1}^{m} f_j^{k+1}) = \frac{1}{w} \sum_{i=1}^{d} h_i$$

where $w = w(f_1) \cdot \ldots \cdot w(f_k) \cdot m_{j_{k+1}} = w(f_1 \cdot \ldots \cdot f_k \cdot f_{k+1})$, $h_1 < \cdots < h_d$ with each $h_i \in D_n$, and $d \leq n_{j_1} + \cdots + n_{j_{k+1}} - k$. This completes the proof of the inductive step and the proof of the proposition. $\square$

Corollary 3.9. Let $f \in M$ with $w(f) < m_j$. Then the functional f can be written in the form $f = \frac{1}{w(f)} \sum_{i=1}^{d} h_i$ with $h_i \in D$, $h_1 < \cdots < h_d$ and $d < n_{j-1} \log_2(m_j)$.

Proof. Let $f = f_1 \cdot \ldots \cdot f_k$ with $f_i \in K$, $w(f_i) = m_{j_i}$, $i = 1, \ldots, k$. Then $m_j > w(f) = w(f_1) \cdot \ldots \cdot w(f_k) \geq 2^k$ and hence $k < \log_2(m_j)$.

Since $j_i \leq j - 1$ for each i, from Proposition 3.8 the functional f takes the form $f = \frac{1}{w(f)} \sum_{i=1}^{d} h_i$ with $d \leq n_{j_1} + \ldots + n_{j_k} - (k-1) < n_{j-1} \log_2(m_j)$. $\square$

Corollary 3.10. Let $f \in M$, $f = f_1 \cdot \ldots \cdot f_k$ with $f_i \in K$, such that $w(f_i) < m_j$ for $i = 1, \ldots, k$ and $w(f) < m_j^2$. Then the functional f takes the form $f = \frac{1}{w(f)} \sum_{i=1}^{d} h_i$ with $h_i \in D$, $h_1 < \cdots < h_d$ and $d < n_{j-1} \log_2(m_j^2)$.

Proof. The proof is almost identical to that of Corollary 3.9, so we omit it. $\square$

Definition 3.11. For $f \in D$ we call tree of f (or tree corresponding to the analysis of f) a family of functionals $T_f = (f_a)_{a \in \mathcal{A}}$ indexed by a finite tree $\mathcal{A}$, with each $f_a \in D$, such that the following conditions are fulfilled:

- (i) The tree $\mathcal{A}$ has a unique root $0 \in \mathcal{A}$ and $f_0 = f$.
- (ii) If a is a maximal element of the tree $\mathcal{A}$ then $f_a \in G$. In this case we say that f_a is of type 0 with weight $w(f_a) = 1$.
- (iii) For every non-maximal $a \in \mathcal{A}$, denoting by S_a the set of immediate successors of a in the tree $\mathcal{A}$, $S_a = \{\beta_1, \cdots, \beta_d\}$, one of the following holds:
  - (a) $f_{\beta_1} < \cdots < f_{\beta_d}$ and $f = \frac{1}{w(f_a)} \sum_{i=1}^{d} f_{\beta_i}$ where $w(f_a) = m_{j_1} \cdot \ldots \cdot m_{j_r}$ and $d \leq n_{j_1} + \cdots + n_{j_r}$. In this case we say that f_a is of type I with weight $w(f_a)$.
  - (b) There exists a family $(\lambda_{\beta_i})_{i=1}^{d}$ of positive rationals with $\sum_{i=1}^{d} \lambda_{\beta_i} = 1$ such that $f_a = \sum_{i=1}^{d} \lambda_{\beta_i} f_{\beta_i}$ and for each i, $\operatorname{ran} f_{\beta_i} \subset \operatorname{ran} f_a$ and f_{β_i} is either of type I or of type 0. In this case we say that f_a is of type II.

Remark 3.12. Every $f \in D$ admits a tree (not necessarily unique). Indeed, it can be shown that each $f \in D_n$ admits a tree, using induction on n and applying Proposition 3.8 in each inductive step.

Proposition 3.13. The Banach algebra $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ of diagonal operators of $\mathfrak{X}_D$ with respect to the basis $(e_n)_{n \in \mathbb{N}}$, is isometric to the dual space $\mathfrak{X}_D^*$.

Proof. Since the norming set D of the space $\mathfrak{X}_D$ contains the characteristic functions $\pm\chi_{\{1,\ldots,n\}}$ for every n (this is the reason we have included the set G in the norming set D) i.e. $\pm \sum_{i=1}^{n} e_i^* \in D$ for every n, condition (3)(a) of Theorem 2.4 is satisfied with constant $C_1 = 1$. The norming set D is closed under pointwise products, i.e. $D \cdot D \subset D$, hence condition (3)(b) of Theorem 2.4 is also satisfied with constant $C_2 = 1$. Theorem 2.4 entails that the space of diagonal operators $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is isometric to the dual space $\mathfrak{X}_D^*$. $\square$

4. The basic inequality and exact pairs

This section is mainly devoted to the statement and the proof of the basic inequality (Proposition 4.8). For this we follow the standard method which has been used in earlier works (i.e. [2], [3], [5], [6], [9]). The general scheme for this method goes as follows. We first define an auxiliary space which is a mixed Tsirelson space and the basic inequality shows that the action of every $f \in D$ on an average of a RIS is dominated by the action of a functional g on the corresponding average of the basis of the auxiliary space plus a small error. This provides, among others, the upper estimate of the norm of averages of a RIS. We show that the space $\mathfrak{X}_D$ is

of codimension 1 in $\mathfrak{X}_D^{**}$ hence is quasireflexive. Finally, we define the exact pairs, a key ingredient for the definition of dependent sequences.

Definition 4.1. Let $C \geq 1$, $\varepsilon > 0$. A block sequence $(x_k)_k$ in $\mathfrak{X}_D$ is said to be a (C, ε) Rapidly Increasing Sequence (RIS) if there exists a strictly increasing sequence $(j_k)_k$ of integers such that the following conditions are satisfied:

(i) $\|x_k\| \leq C$ and $\|x_k\|_G \leq \varepsilon$ for each k.

(ii) $\frac{1}{m_{j_1}} \leq \varepsilon$ and $\# \operatorname{supp}(x_k) \cdot \frac{1}{m_{j_{k+1}}} \leq \varepsilon$ for all k.

(iii) For every k and $f \in D$ with $w(f) < m_{j_k}$ it holds that $|f(x_k)| \leq \frac{C}{w(f)}$.

We call the sequence of integers $(j_k)_k$, the associated sequence of the RIS $(x_k)_k$.

Definition 4.2 (The auxiliary space)**.** Let W be the minimal subset of $c_{00}(\mathbb{N})$ such that

(i) It contains $\pm e_n^*$, $n \in \mathbb{N}$.

(ii) It is closed under the $(\mathcal{A}_{2n_j}, \frac{1}{m_j})$ operation for every j.

(iii) It is rationally convex.

We also define W' as the minimal subset of $c_{00}(\mathbb{N})$ satisfying the above conditions (i), (ii).

Remark 4.3. It is easily seen that a subset of $c_{00}(\mathbb{N})$ which is closed under the $(\mathcal{A}_n, \theta)$ and $(\mathcal{A}_{n'}, \theta')$ operations, it is also closed under the the $(\mathcal{A}_{nn'}, \theta\theta')$ operation. It follows that the set W is closed under the $(\mathcal{A}_{(2n_{j_1})\cdot\ldots\cdot(2n_{j_k})}, \frac{1}{m_{j_1}\cdot\ldots\cdot m_{j_k}})$ operation, for every $j_1, \ldots, j_k \in \mathbb{N}$ (not necessarily distinct). Since $\sum_{i=1}^{k} 2n_{j_i} \leq \prod_{i=1}^{k} 2n_{j_i}$ we get that the set W (and the set W' also) is closed under the $(\mathcal{A}_{2n_{j_1}+\cdots+2n_{j_k}}, \frac{1}{m_{j_1}\cdot\ldots\cdot m_{j_k}})$ operation.

Remark 4.4. The trees for functionals $g \in W$, are defined in a similar manner as the corresponding ones for $g \in D$ (Definition 3.11), the only difference being that the functionals corresponding to maximal elements are of the form $\pm e_r^*$. For $f \in W'$ the trees are defined as those for $g \in W$, the only difference being that we require that no functionals of type II appear.

Lemma 4.5. *Let $g \in W$ with a tree $(g_a)_{a\in\mathcal{A}}$. Then the functional g is a rational convex combination $g = \sum_{i\in I} \lambda^i g^i$, such that for each $i \in I$, $g^i \in W'$, the functional g^i has a tree $(g_a^i)_{a\in\mathcal{A}^i}$ and there exists an order preserving map $\Phi^i : \mathcal{A}^i \to \mathcal{A}$ satisfying the following.*

(i) For every maximal node $a \in \mathcal{A}^i$, $\Phi^i(a)$ is a maximal node of $\mathcal{A}$ and $g_a^i = g_{\Phi^i(a)}$.

(ii) For every non-maximal $a \in \mathcal{A}^i$, the functionals g_a^i, $g_{\Phi^i(a)}$ are of type I with $w(g_a^i) = w(g_{\Phi^i(a)})$ and $\#S_a^i = \#S_{\Phi^i(a)}$, where S_a^i denotes the set of immediate successors of $a \in \mathcal{A}^i$ and S_γ is the set of immediate successors of a $\gamma \in \mathcal{A}$.

(iii) If the functional g is weighted then $w(g^i) = w(g)$.

Proof. We shall prove, using backward induction, that for each $a \in \mathcal{A}$ the functional g_a with the tree $(g_a)_{a\in\mathcal{A}^{\succeq\beta}}$, where $\mathcal{A}^{\succeq\beta} = \{a \in \mathcal{A} : a \succeq \beta\}$, satisfies the conclusion of the lemma. This will finish the proof of the lemma, since $g = g_0$, where $0 \in \mathcal{A}$ denotes the unique root of the tree $\mathcal{A}$.

The first inductive step concerns $a \in \mathcal{A}$ which is maximal. In this case, setting $I_a = \{1\}$, $\lambda^1 = 1$, $\mathcal{A}^1 = \{a\}$ and $\Phi^1(a) = a$, the required conditions are obviously satisfied.

Let us pass to the general inductive step. We consider $a \in \mathcal{A}$ which is non-maximal, $S_a = \{\beta_1, \dots, \beta_d\}$ and we assume that for each $k = 1, \dots, d$, the functional g_{β_k} takes the form $g_{\beta_k} = \sum\limits_{i \in I_{\beta_k}} \lambda^i_{\beta_k} g^i_{\beta_k}$ and each $g^i_{\beta_k} \in W'$ has a tree $(g^i_{\beta_k,\gamma})_{\gamma \in \mathcal{A}^i_{\beta_k}}$ such that there exists an order preserving map $\Phi^i_{\beta_k} : \mathcal{A}^i_{\beta_k} \to \mathcal{A}^{\succeq \beta_k}$ satisfying conditions (i), (ii), (iii). We distinguish two cases.

Case 1. The functional g_a is of type I, $g_a = \frac{1}{w(g_a)}(g_{\beta_1} + \cdots + g_{\beta_d})$.
We set $I_a = I_{\beta_1} \times \cdots \times I_{\beta_d}$ and for each $i = (i_1, \dots, i_d) \in I_a$ we define $\lambda^i_a = \lambda^{i_1}_{\beta_1} \cdot \dots \cdot \lambda^{i_d}_{\beta_d}$ and $g^i_a = \frac{1}{w(g_a)}(g^{i_1}_{\beta_1} + \cdots + g^{i_d}_{\beta_d})$. It is evident that g_a equals to the convex combination $\sum\limits_{i \in I_a} \lambda^i_a g^i_a$. For each $i = (i_1, \dots, i_d) \in I_a$ we define the tree $\mathcal{A}^i_a$ as the disjoint union $\mathcal{A}^i_a = \{a\} \cup \bigcup\limits_{k=1}^{d} \mathcal{A}^{i_k}_{\beta_k}$ with its ordering defined by the rule $\delta_1 \preceq \delta_2$ if and only if $\delta_1 = a$ or if there exists $k \in \{1, \dots, d\}$ such that $\delta_1, \delta_2 \in \mathcal{A}^{i_k}_{\beta_k}$ and $\delta_1 \preceq \delta_2$ in the ordering of $\mathcal{A}^{i_k}_{\beta_k}$. We also define the order preserving map $\Phi^i_a : \mathcal{A}^i_a \to \mathcal{A}^{\succeq a}$ as follows; $\Phi^i_a(a) = a$ and $\Phi^i_a(\gamma) = \Phi^{i_k}_{\beta_k}(\gamma)$ for $\gamma \in \mathcal{A}^{i_k}_{\beta_k}$. It is trivial to observe that conditions (i), (ii), (iii) are satisfied.

Case 2. The functional g_a is of type II, $g_a = \sum\limits_{k=1}^{d} \lambda_{\beta_k} g_{\beta_k}$.

Then, using our inductive hypothesis, we get that $g_a = \sum\limits_{k=1}^{d} \sum\limits_{i \in I_{\beta_k}} (\lambda_{\beta_k} \lambda^i_{\beta_k}) g^i_{\beta_k}$ as a convex combination, while the required conditions (i), (ii), (iii) are obviously satisfied.

The proof of the Lemma is complete. □

Lemma 4.6. Let $g \in W$, $g = \frac{1}{w(g)} \sum\limits_{i=1}^{d} g_i$ where $w(g) = m_{j_1} \cdot \dots \cdot m_{j_r}$ and $g_1 < \cdots < g_d$ with $g_i \in W$ and $d \leq 2n_{j_1} + \cdots + 2n_{j_r}$. Let also $j \in \mathbb{N}$. Then

$$|g(\frac{1}{n_j} \sum_{k=1}^{n_j} e_k)| \leq \begin{cases} \frac{2}{w(g) m_j} & \text{if } w(g) < m_j \\ \frac{1}{w(g)} & \text{if } w(g) \geq m_j. \end{cases}$$

Proof. From Lemma 4.5, we may assume that $g \in W'$ and $g_i \in W'$ for each $i = 1, \dots, d$. We start with the following claim.

Claim. For every $f \in W'$, $\#\{k : \ |f(e_k)| > \frac{1}{m_j}\} \leq (2n_{j-1})^{\log_2(m_j)-1}$.

Proof of the claim. We may select a family of functionals $(f_a)_{a \in \mathcal{A}}$ indexed by a finite tree $\mathcal{A}$, with $f_a \in W$, such that

(i) The tree $\mathcal{A}$ has a unique root $0 \in \mathcal{A}$ and $f_0 = f$.
(ii) For every maximal node $a \in \mathcal{A}$, $f_a = \pm e^*_n$.
(iii) For every non-maximal node $a \in \mathcal{A}$, there exists $j_a \in \mathbb{N}$ such that $f_a = \frac{1}{m_{j_a}}(f_{\beta_1} + \cdots + f_{\beta_d})$ and $d \leq 2n_{j_a}$ where $S_a = \{\beta_1, \dots, \beta_d\}$ is the set of immediate successors of a in $\mathcal{A}$ and $f_{\beta_1} < \cdots < f_{\beta_d}$.

We may assume that $f(e_k) > \frac{1}{m_j}$ for every $k \in \mathrm{supp}(f)$. Since $w(f_a) \geq m_1 \geq 2$ for every non-maximal $a \in \mathcal{A}$ it follows that the cardinality of every branch of the tree $\mathcal{A}$ is less than $\log_2(m_j)$. An easy inductive argument implies that $\#\,\mathrm{supp}(f) \leq (2n_{j-1})^{\log_2(m_j)-1}$. $\square$

Let now $g \in W'$. The case $w(g) \geq m_j$ is obvious. Assume that $w(g) < m_j$. Then, as in the proof of Corollary 3.9, it follows that $d \leq 2n_{j-1}\log_2(m_j)$. For $i = 1, \dots, d$, set $L_i = \{k : \ |g_i(e_k)| > \frac{1}{m_j}\}$ and $L = \bigcup_{i=1}^{d} L_i$. From the claim above we get that $\#L_i \leq (2n_{j-1})^{\log_2(m_j)-1}$ for each i, thus $\#L \leq (2n_{j-1})^{\log_2(m_j)}\log_2(m_j)$. Therefore, splitting the functional g as $g = g|_L + g|_{\mathbb{N}\setminus L}$ we get that

$$\begin{aligned} |g(\frac{1}{n_j}\sum_{k=1}^{n_j} e_k)| &\leq \frac{1}{w(g)}\frac{1}{n_j}\#L + \frac{1}{w(g)}\frac{1}{m_j} \\ &\leq \frac{1}{w(g)}\Big(\frac{1}{n_j}(2n_{j-1})^{\log_2(m_j)}\log_2(m_j) + \frac{1}{m_j}\Big) \leq \frac{2}{w(g)\cdot m_j}. \end{aligned}$$

$\square$

Lemma 4.7. Let $g \in W$ and suppose that the functional g admits a tree $(g_a)_{a\in\mathcal{A}}$ with the following property. For every $a \in \mathcal{A}$ such that g_a is of type I with $w(g_a) < m_{j_0}^2$, the cardinality of the set S_a of immediate successors of a in $\mathcal{A}$ satisfies $\#S_a \leq m_{j_0}^2 Q_{j_0}$ $\big($recall from Definition 3.1 that $Q_{j_0} = \big(n_{j_0-1}\cdot\log_2(m_{j_0})\big)^{\log_2(m_{j_0})}\big)$. Then

$$|g(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} e_k)| \leq \frac{2}{m_{j_0}^2}.$$

Proof. It is enough to prove the statement for $g \in W'$, since from Lemma 4.5 the functional g takes the form $g = \sum_{i\in I}\lambda^i g^i$ as a convex combination, with each $g^i \in W'$ and such that each g^i satisfies the assumption of the statement.

Let $g \in W'$ satisfying the assumption of the statement of the lemma. We set

$$B_1 = \{k : \ |g(e_k)| > \frac{1}{m_{j_0}^2}\} \qquad B_2 = \{k : \ |g(e_k)| \leq \frac{1}{m_{j_0}^2}\}$$

and we consider the functionals $g_1 = g_{|B_1}$ and $g_2 = g_{|B_2}$. Then obviously

$$|g_2(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} e_k)| \leq \frac{1}{m_{j_0}^2}. \tag{2}$$

The property of the statement remains valid for the functional g_1; let $(g_a)_{a\in\mathcal{A}}$ be a tree of the functional g_1, satisfying the aforementioned property. Note that no functionals of type II appear in this tree, since $g_1 \in W'$ (see Remark 4.4). The fact that $|g_1(e_k)| > \frac{1}{m_{j_0}^2}$ for every $k \in \mathrm{supp}\, g_1$ implies that every branch of the tree $\mathcal{A}$ has length at most $\log_2(m_{j_0}^2)$. Since also $w(g_a) < m_{j_0}^2$ for every $a \in \mathcal{A}$, our assumption entails that each non-maximal $a \in \mathcal{A}$ has at most $m_{j_0}^2 Q_{j_0}$ immediate successors. Thus $\#\,\mathrm{supp}\, g_1 \leq (m_{j_0}^2 Q_{j_0})^{\log_2(m_{j_0}^2)}$. Using the growth condition concerning the

sequence $(n_j)_j$ (see Definition 3.1) we derive that

$$(3) \qquad |g_1(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} e_k)| \leq \frac{1}{n_{j_0}} \cdot \#\operatorname{supp} g_1 \leq \frac{1}{n_{j_0}} \cdot m_{j_0}^{4\log_2(m_{j_0})} Q_{j_0}^{2\log_2(m_{j_0})} \leq \frac{1}{m_{j_0}^2}.$$

From (2), (3) we conclude that $|g(\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}} e_k)| \leq \frac{2}{m_{j_0}^2}$. $\square$

Proposition 4.8 (basic inequality)**.** Let $(x_k)_{k=1}^{n_{j_0}}$ be a (C,ε) RIS in $\mathfrak{X}_D$ with associated sequence $(j_k)_{k=1}^{n_{j_0}}$. Then for every $f \in D$ there exists a functional $g \in W$ such that the following conditions are fulfilled.

(1) If f is of type I then either $w(g) = w(f)$ or $g = e_r^*$ or $g = 0$.

(2) $|f(\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}} x_k)| \leq C\big(g(\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}} e_k) + \varepsilon\big)$.

If we additionally assume that $j_0 < j_1$ and that for every subinterval I of the interval $\{1, 2, \ldots, n_{j_0}\}$ with $\#(I) \geq m_{j_0}^2 Q_{j_0}$ and for every functional $h \in K^{j_0} \cdot D$ it holds that $|h(\sum\limits_{k\in I} x_k)| \leq C \cdot \varepsilon \cdot \#I$, then the functional $g \in W$ may selected to satisfy in addition the following property:

(3) The functional g admits a tree $(g_\beta)_{\beta\in\mathcal{B}}$ with the property that for every $\beta \in \mathcal{B}$ with g_β of type I and $w(g_\beta) < m_{j_0}^2$, the node β has at most $m_{j_0}^2 Q_{j_0}$ immediate successors.

Proof. We begin with the proof of the first part of the proposition (without the additional assumption). We fix a tree $(f_a)_{a\in\mathcal{A}}$ of the functional f (Definition 3.11). Using backward induction we shall define, for every $a \in \mathcal{A}$ and every subinterval J of $\{1, 2, \ldots, n_{j_0}\}$, a functional $g_a^J \in W$ such that the following conditions are satisfied.

(i) If f_a of type I then either $g_a^J = e_r^*$ or $g_a^J = 0$ or g_a^J is of type I and takes the form $g_a^J = \frac{1}{w(f_a)}(\sum\limits_{i=1}^{d} g_{\beta_i}^{J_i} + \sum\limits_{k\in J_0} e_k^*)$ where $S_a = \{\beta_1, \ldots, \beta_d\}$, each J_i for $1 \leq i \leq d$ is a subinterval of J, $J_0 \subset J$ and the sets $J_0, J_1, \ldots, J_d$ are pairwise disjoint.

(ii) $|f_a(\sum\limits_{k\in J} x_k)| \leq C\big(g_a^J(\sum\limits_{k\in J} e_k) + \varepsilon \cdot \#J\big)$.

(iii) The functional g_a^J has nonnegative coordinates and $\operatorname{supp} g_a^J \subset J$.

When the inductive construction is completed, the functional $g = g_0^{J_0}$, where $0 \in A$ is the root of the tree $\mathcal{A}$ and $J_0 = \{1, \ldots, n_{j_0}\}$, satisfies the conclusion of the proposition.

The first inductive step concerns $a \in \mathcal{A}$ which are maximal. Then $f_a \in G$. We set $g_a^J = 0$ for every subinterval J. Since $\|x_k\|_G \leq \varepsilon$ for each k (condition (i) of Definition 4.1) it follows that $|f_a(\sum\limits_{k\in J} x_k)| \leq \sum\limits_{k\in J} |f_a(x_k)| \leq \varepsilon\cdot\#J$ and thus condition (ii) is satisfied.

Let us pass to the general inductive step. We distinguish two cases.

Case 1. The functional f_a is of type II, $f_a = \sum\limits_{i=1}^{d} \lambda_{\beta_i} f_{\beta_i}$.

For every subinterval J we set $g_a^J = \sum\limits_{i=1}^{d} \lambda_{\beta_i} g_{\beta_i}^J$. Then, using the inductive hypothesis

we get that

$$\begin{aligned}|f_a(\sum_{k\in J}x_k)| &\leq \sum_{i=1}^{d}\lambda_{\beta_i}|f_{\beta_i}(\sum_{k\in J}x_k)| \leq \sum_{i=1}^{d}\lambda_{\beta_i}C\big(g^J_{\beta_i}(\sum_{k\in J}e_k)+\varepsilon\cdot\#J\big)\\ &= C\big(g^J_a(\sum_{k\in J}e_k)+\varepsilon\cdot\#J\big).\end{aligned}$$

Case 2. The functional f_a is of type I, $f_a=\frac{1}{w(f_a)}(f_{\beta_1}+f_{\beta_2}+\cdots+f_{\beta_d})$ (where $w(f_a)=m_{j_1^a}\cdot\ldots\cdot m_{j_r^a}$ and $d\leq n_{j_1^a}+\cdots+n_{j_r^a}$ for some $j_1^a,\ldots,j_r^a\in\mathbb{N}$).
Fix J a subinterval of $\{1,\ldots,n_{j_0}\}$. We distinguish three subcases.

Subcase 2a. $m_{j_{k_0}}\leq w(f_a)<m_{j_{k_0+1}}$ for some $k_0\in J$.
Then for $k\in J$ with $k<k_0$ we have that $m_{j_{k+1}}\leq m_{j_{k_0}}\leq w(f_a)$ and thus, using conditions (i), (ii) of Definition 4.1, we get that

$$|f_a(x_k)|\leq\frac{1}{w(f_a)}\|x_k\|_{\ell_1}\leq\frac{1}{m_{j_{k+1}}}\cdot C\cdot\#\operatorname{supp}(x_k)\leq C\cdot\varepsilon.$$

For $k\in J$ with $k>k_0$, using conditions (ii), (iii) of Definition 4.1, we get that

$$|f_a(x_k)|\leq\frac{C}{w(f_a)}\leq\frac{C}{m_{j_1}}\leq C\cdot\varepsilon.$$

If $\operatorname{ran}(f_a)\cap\operatorname{ran}(x_{k_0})\neq\emptyset$ we set $g^J_a=e^*_{k_0}$, otherwise we set $g^J_a=0$. The inductive conditions are easily established.

Subcase 2b. $m_{j_{k+1}}\leq w(f)$ for every $k\in J$.
Then, as before, $|f_a(x_k)|\leq C\cdot\varepsilon$ for every $k\in J$ and we set $g^J_a=0$.

Subcase 2c. $w(f_a)<m_{j_k}$ for all $k\in J$.
Let $E_i=\operatorname{ran}f_{\beta_i}$ for $i=1,\ldots,d$. We set

$$J_0=\{k\in J:\ \operatorname{ran}(x_k)\text{ intersects at least two }E_i,\ i=1,\ldots,d\}$$

and for $i=1,\ldots,d$ we set

$$J_i=\{k\in J\setminus J_0:\ \operatorname{ran}(x_k)\text{ intersects }E_i\}.$$

We observe that $\#J_0\leq d$ and that each J_i is a subinterval of J. From our inductive hypothesis the functionals $(g^{J_i}_{\beta_i})_{i=1}^d$ have been defined satisfying conditions (i), (ii), (iii). The family $\{J_i:\ i=1,\ldots,d\}\cup\{\{k\}:\ k\in J_0\}$ consists of pairwise disjoint intervals while its cardinality does not exceed $2d$. We define

$$g^J_a=\frac{1}{w(f_a)}(\sum_{i=1}^{d}g^{J_i}_{\beta_i}+\sum_{k\in J_0}e^*_k).$$

From our definitions it follows that $g^J_a\in W$ (see Remark 4.3). Observe, for later use, the following. If $w(f_a)<m^2_{j_0}$ and $f_a\notin K^{j_0}\cdot D$ then Corollary 3.10 entails that $d\leq n_{j_0-1}\log_2(m^2_{j_0})$ and hence $2d\leq 4n_{j_0-1}\log_2(m_{j_0})\leq m^2_{j_0}Q_{j_0}$.

Since $w(f_a)<m_{j_k}$ for every $k\in J$, condition (iii) of Definition 4.1 implies that $|f_a(x_k)|\leq\frac{C}{w(f_a)}$ for every $k\in J$. From this and from our inductive hypotheses we

get that

$$\begin{aligned}|f_a(\sum_{k\in J}x_k)| &\leq \sum_{k\in J_0}|f_a(x_k)| + \frac{1}{w(f_a)}\sum_{i=1}^{d}|f_{\beta_i}(\sum_{k\in J_i}x_k)| \\ &\leq \sum_{k\in J_0}\frac{C}{w(f_a)} + \frac{1}{w(f_a)}\sum_{i=1}^{d}C\big(g_{\beta_i}^{J_i}(\sum_{k\in J_i}e_k) + \varepsilon\cdot \#J_i\big) \\ &\leq C\big(g_a(\sum_{k\in J}e_k) + \varepsilon\cdot\#J\big).\end{aligned}$$

This completes the proof of the general inductive step and finishes the proof of the first part of the proposition.

Next, we proceed with the proof of the second part of the proposition, where the additional assumption is made. Then in addition to conditions (i), (ii), (iii) we require the following.

(iv) If g_a^J is of type I with $w(g_a^J) < m_{j_0}^2$ then $\#\{i:\ g_{\beta_i}^{J_i}\neq 0\} + \#J_0 \leq m_{j_0}^2 Q_{j_0}$ (where the notation comes from condition (i)).

The procedure remains the same for the first inductive step and for Case 1, Subcase 2a and Subcase 2b in the general inductive step. The difference concerns Subcase 2c (i.e. when $w(f_a) < m_{j_k}$ for all $k \in J$) where we distinguish two subsubcases.

Subsubcase 2cA. $f_a \in K^{j_0}\cdot D$ and $\#J \geq m_{j_0}^2 Q_{j_0}$.
We set $g_a^J = 0$ and our additional assumption yields condition (ii).

Subsubcase 2cB. $f_a \notin K^{j_0}\cdot D$ or $\#J < m_{j_0}^2 Q_{j_0}$.
We proceed exactly as in the proof of Subcase 2c in the first part of the proposition.

We have to examine condition (iv). Observe that, if g_a^J is of type I with $w(g_a^J) < m_{j_0}^2$ there are two cases. Either $f_a \notin K^{j_0}\cdot D$ and $w(f_a) = w(g_a^J) < m_{j_0}^2$, in which case (see the proof of Subcase 2c) the sum in the left side of (iv) does not exceed $2d$ which is at most $m_{j_0}^2 Q_{j_0}$, or $\#J < m_{j_0}^2 Q_{j_0}$, in which case the same upper bound is derived from the fact that $\operatorname{supp} g_a^J \subset J$.

This completes the proof of the general inductive step and the proof of the second part of the proposition. $\square$

Corollary 4.9. Let $(x_k)_{k=1}^{n_{j_0}}$ be a (C,ε) RIS with $\varepsilon \leq \frac{1}{m_{j_0}^2}$. Then for $f \in D$ of type I we have that

$$|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}}x_k)| \leq \begin{cases} \frac{3C}{w(f)m_{j_0}} & \text{if } w(f) < m_{j_0} \\ C(\frac{1}{w(f)} + \frac{1}{m_{j_0}^2}) & \text{if } w(f) \geq m_{j_0}.\end{cases}$$

In particular $\|\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}}x_k\| \leq \frac{2C}{m_{j_0}}$.

Proof. From the basic inequality (Proposition 4.8) there exists $g \in W$ with either $w(g) = w(f)$ or $g = e_r^*$ or $g = 0$, such that $|f(\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}}x_k)| \leq C\big(g(\frac{1}{n_{j_0}}\sum\limits_{k=1}^{n_{j_0}}e_k) + \varepsilon\big)$.

From Lemma 4.6 we get that if $w(f) < m_{j_0}$ then $|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k)| \leq C(\frac{2}{w(f)m_{j_0}} + \frac{1}{m_{j_0}^2}) \leq \frac{3C}{w(f)m_{j_0}}$, while if $w(f) \geq m_{j_0}$ then $|f(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k)| \leq C(\frac{1}{w(f)} + \frac{1}{m_{j_0}^2})$. $\square$

Definition 4.10. A vector $x \in \mathfrak{X}_D$ is said to be a $C - \ell_1^k$ average if x takes the form $x = \frac{1}{k}\sum_{i=1}^{k} x_i$, with $\|x_i\| \leq C$ for each i, $x_1 < \cdots < x_k$ and $\|x\| \geq 1$.

Lemma 4.11. Let Y be a block subspace of $\mathfrak{X}_D$ and $k \in \mathbb{N}$. Then there exists a vector $x \in Y$ which is a $2 - \ell_1^k$ average.

For a proof we refer to [7] Lemma II.22.

Lemma 4.12. If x is a $C - \ell_1^k$ average, $d \leq k$ and $E_1 < \cdots < E_d$ is any sequence of intervals, then $\sum_{i=1}^{d} \|E_i x\| \leq C(1 + \frac{2d}{k})$.

For a proof we refer to [7] Lemma II.23.

Remark 4.13. (i) If x is a $C - \ell_1^{n_j}$ average and f is of type I with $w(f) < m_j$, then from Lemma 4.12 and Corollary 3.9 we get that $|f(x)| \leq \frac{2C}{w(f)}$.
(ii) Suppose that $(x_k)_{k\in\mathbb{N}}$ is a block sequence in $\mathfrak{X}_D$, such that each x_k is a $C - \ell_1^{n_{j_k}}$ average for an increasing sequence $(j_k)_{k\in\mathbb{N}}$ and $\|x_k\|_G \leq \varepsilon$ for all k. From (i), if $f \in D$ with $w(f) < m_{j_k}$ then $|f(x_k)| \leq \frac{2C}{w(f)}$. Thus, we may easily select a subsequence of $(x_k)_{k\in\mathbb{N}}$ which is a $(2C, \varepsilon)$ RIS.

Proposition 4.14. The space $\mathfrak{X}_D$ is a strictly singular extension of Y_G (i.e. the identity operator $I : \mathfrak{X}_D \to Y_G$, where Y_G is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_G)$, is strictly singular).

Proof. Assume the contrary. Then we can find a block subspace Z of $\mathfrak{X}_D$ and $\varepsilon > 0$ such that $\|z\|_G \geq \varepsilon\|z\|$ for every $z \in Z$. Pick $(z_n)_{n\in\mathbb{N}}$, $(z_n^*)_{n\in\mathbb{N}}$ two normalized block sequences in Z, $\mathfrak{X}_D^*$ respectively, such that $\operatorname{ran} z_n = \operatorname{ran} z_n^*$ and $z_n^*(z_n) = 1$. Passing to subsequences we may assume that the sequence $(z_n)_{n\in\mathbb{N}}$ is weakly Cauchy in Y_G, hence the sequence of its successive differences, i.e. the sequence $(x_n)_{n\in\mathbb{N}}$ defined by the rule $x_n = z_{2n-1} - z_{2n}$ is weakly null in Y_G. Thus $\lim_n \chi_{\mathbb{N}}(x_n) = 0$. Passing to further subsequences we may assume that $\sum_{n=1}^{\infty} |\chi_{\mathbb{N}}(x_n)| < \frac{1}{2}$. Then $\|\sum_{i=1}^{k} x_i\|_G < \frac{9}{2}$ for every $k \in \mathbb{N}$ (see the proof of Lemma 6.3).

Let now $j \in \mathbb{N}$. We set $x = \frac{1}{n_{2j}}\sum_{i=1}^{n_{2j}} x_i$ and $x^* = \frac{1}{m_{2j}}\sum_{i=1}^{n_{2j}} z_{2i-1}^*$. Then $\|x^*\| \leq 1$ (see Remark 3.5), hence

$$\frac{1}{n_{2j}} \cdot \frac{9}{2} \geq \|x\|_G \geq \varepsilon \cdot \|x\| \geq \varepsilon \cdot x^*(x) = \varepsilon \cdot \frac{1}{m_{2j}}.$$

Since this happens for every j, for large j we derive a contradiction. $\square$

Definition 4.15. Let $C \geq 1$ and $j \in \mathbb{N}$. A pair (x, x^*) is said to be a $(C, 2j)$ exact pair, provided that

(i) $x^* \in K$ with $w(x^*) = m_{2j}$ (i.e. $x^* \in K^{2j}$).

(ii) $\|x\|_G \leq \frac{1}{m_{2j}^2}$, and for every $f \in D$ of type I, if $w(f) < m_{2j}$ then $|f(x)| \leq \frac{3C}{w(f)}$, while if $w(f) \geq m_{2j}$ then $|f(x)| \leq C(\frac{m_{2j}}{w(f)} + \frac{1}{m_{2j}})$ (in particular $\|x\| \leq 2C$).

(iii) $x^*(x) = 1$ and $\operatorname{ran} x^* = \operatorname{ran} x$.

Lemma 4.16. For every block subspace Z of $\mathfrak{X}_D$ and every $j \in \mathbb{N}$ there exists a $(4, 2j)$ exact pair (z, z^*) with $z \in Z$.

Proof. We set $\varepsilon = \frac{1}{m_{2j}^3}$. Using Proposition 4.14, we may select a block subspace Z' of Z such that the restriction of the identity operator $I : \mathfrak{X}_D \to Y_G$ on Z' has norm less than $\frac{\varepsilon}{2}$. From Lemma 4.11 we may select a block sequence $(z_k)_{k\in\mathbb{N}}$ in Z' with each z_k being a $2 - \ell_1^{n_{j_k}}$ average for an increasing sequence $(j_k)_{k\in\mathbb{N}}$. Then $\|z_k\|_G \leq \varepsilon$ for all k. From Remark 4.13 we may assume, passing to a subsequence, that $(z_k)_{k=1}^{n_{2j}}$ is a $(4, \varepsilon)$ RIS. For each k we select $z_k^* \in D$ with $\operatorname{ran}(z_k^*) = \operatorname{ran}(z_k)$ such that $z_k^*(z_k) \geq 1$.

We set

$$x = \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k \quad \text{and} \quad z^* = \frac{1}{m_{2j}} \sum_{k=1}^{n_{2j}} z_k^*.$$

From Corollary 4.9 we get that $\frac{1}{m_{2j}} \leq z^*(x) \leq \|x\| \leq \frac{8}{m_{2j}}$, thus we may select $\frac{1}{8} \leq \theta \leq 1$ such that $z^*(\theta m_{2j} x) = 1$. We set $z = \theta m_{2j} x$. From the fact that $\|z\|_G \leq m_{2j} \cdot \varepsilon = \frac{1}{m_{2j}^2}$, and using Corollary 4.9, we easily establish condition (ii) of Definition 4.15. Hence (z, z^*) is a $(4, 2j)$ exact pair. $\square$

Remark 4.17. Exact pairs are one of the fundamental ingredients to show that the space $\mathfrak{X}_D$ is HI. This property will be proved in the next section. A rather direct consequence of the previous lemma is that $\ell_1(\mathbb{N})$ is not embedded in the space $\mathfrak{X}_D$. To see this, observe that Lemma 4.16 yields that for every $j \in \mathbb{N}$ there exists a normalized finite block sequence $(w_k)_{k=1}^{n_{2j}}$ in Z such that $\|\frac{w_1 + w_2 + \cdots + w_{n_{2j}}}{n_{2j}}\| \leq \frac{4}{m_{2j}}$.

Proposition 4.18. The space $\mathfrak{X}_D^*$ is the closed lineal span of the pointwise closure of G, i.e. $\mathfrak{X}_D^* = \overline{\operatorname{span}}(\overline{G}^{w^*}) = \overline{\operatorname{span}}(\{e_n^* : n \in \mathbb{N}\} \cup \{\chi_{\mathbb{N}}\})$. Thus the space $\mathfrak{X}_D$ is quasireflexive, with its codimension in the second dual being equal to one.

Proof. Suppose that the space $Z = \overline{\operatorname{span}}(\overline{G}^{w^*})$ is a proper subspace of $\mathfrak{X}_D^*$. Then we may select an $x^* \in \mathfrak{X}_D^* \setminus Z$ with $\|x^*\| = 1$ and an $x^{**} \in \mathfrak{X}_D^{**}$ such that $\|x^{**}\| = 2$, $x^{**}(x^*) = 2$ and $Z \subset \operatorname{Ker} x^{**}$. Since the space $\mathfrak{X}_D$ contains no isomorphic copy of $\ell_1(\mathbb{N})$ (Remark 4.17), from a theorem of Odell and Rosenthal ([17]), we can choose $(x_k)_{k\in\mathbb{N}}$ with $\|x_k\| \leq 2$ such that $x_k \xrightarrow{w^*} x^{**}$. We may assume that $(x_k)_{k\in\mathbb{N}}$ is a block sequence (since $e_n^* \in Z$ for all n) and that $x^*(x_k) > 1$ for each k (since $x^*(x_k) \to x^{**}(x^*) = 2$).

From the fact that $\operatorname{Ext}(B_{Y_G^*}) \subset \overline{G}^{w^*} \subset Z$ we get that the sequence $(x_k)_{k\in\mathbb{N}}$ is weakly null in Y_G, thus we may select a block sequence $(y_k)_{k\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ such that $\|y_k\|_G \to 0$ and such that each y_k is a convex combination of $(x_k)_{k\in\mathbb{N}}$. Passing to a subsequence of $(y_k)_{k\in\mathbb{N}}$, we may assume that $\|y_k\|_G < \varepsilon$ for all k where $\varepsilon = \frac{1}{m_{2j}^3}$ for some fixed j. Notice also that $x^*(y_k) > 1$ for all k. We may construct $(z_k)_{k\in\mathbb{N}}$ a sequence of $2 - \ell_1^{n_{j_k}}$ averages of increasing length with each z_k being an average of $(y_k)_{k\in\mathbb{N}}$. Thus $\|z_k\|_G < \varepsilon$ for all k and passing to a subsequence we may assume

that $(z_k)_{k=1}^{n_{2j}}$ is a $(4, 2j)$ RIS. From Corollary 4.9 we get that $\|z\| \leq \frac{8}{m_{2j}} < 1$, where $z = \frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k$. On the other hand the action of the functional x^* entails that $\|z\| \geq x^*(z) > 1$ (since z is a convex combination of $(y_k)_{k\in\mathbb{N}}$), a contradiction. $\square$

5. Dependent sequences and the HI property of $\mathfrak{X}_D$

In this section we define the dependent sequences which will be used for the proof that the spaces $\mathfrak{X}_D$, $\mathfrak{X}_D^*$ are HI. The main goal is to prove Lemma 5.4 and then to use the basic inequality to obtain upper estimates of the norm for certain averages of block sequences (Proposition 5.5). As a consequence, we obtain that the space $\mathfrak{X}_D$ is Hereditarily Indecomposable.

Definition 5.1. Let $C \geq 1$ and $j \geq 2$. A double sequence $\chi = (x_k, x_k^*)_{k=1}^{n_{2j-1}}$ is said to be a $(C, 2j-1)$ dependent sequence provided there exists a sequence $(j_k)_{k=1}^{n_{2j-1}}$ such that

(i) The sequence $(x_k^*)_{k=1}^{n_{2j-1}}$ is a $2j-1$ special sequence (Definition 3.4) with $w(x_k^*) = m_{2j_k}$ for each k.
(ii) Each (x_k, x_k^*) is a $(C, 2j_k)$ exact pair.

For χ as above we shall denote by ϕ_χ the functional $\phi_\chi = \frac{1}{m_{2j-1}} \sum_{k=1}^{n_{2j-1}} x_k^*$. Observe that $\phi_\chi \in K^{2j-1} \subset D$ (Remark 3.3(iii)), hence $\|\phi_\chi\| \leq 1$.

Remark 5.2. From Definitions 3.4, 4.15 and 5.1 and the growth condition concerning the coding function σ, it follows that when $(x_k, x_k^*)_{k=1}^{n_{2j-1}}$ is a $(C, 2j-1)$ dependent sequence, the sequence $(x_k)_{k=1}^{n_{2j-1}}$ is a $(3C, \frac{1}{m_{2j-1}^2})$ RIS.

Lemma 5.3. Let $f \in D$ and let $\chi = (x_k, x_k^*)_{k=1}^{n_{2j-1}}$ be a $(C, 2j-1)$ dependent sequence. Then for any subinterval I of the interval $\{1, 2, \ldots, n_{2j-1}\}$, we have that

$$|(f \cdot \phi_\chi)\big(\sum_{k\in I} (-1)^{k+1} x_k\big)| \leq \frac{4C}{m_{2j-1}^2} \cdot \#(I) + 2C \cdot Q_{2j-1}.$$

Proof. We fix a tree $(f_a)_{a\in\mathcal{A}}$ of the functional f (Definition 3.11). For each $a \in \mathcal{A}$ we set $w_a = \prod\{w(f_\beta) : \beta \in \mathcal{A},\ \beta \prec a,\ f_\beta \text{ is of type I }\}$. We consider the following subsets of $\mathcal{A}$.

$$\begin{aligned}
\mathcal{D}_0 &= \{a \in \mathcal{A} :\ f_a \text{ is of type 0 and } w_a < m_{2j-1}\ \} \\
\mathcal{D}_1 &= \{a \in \mathcal{A} :\ f_a \text{ is of type I, } w_a < m_{2j-1} \text{ and } w(f_a) \geq m_{2j-1}\ \} \\
\mathcal{D}_2 &= \{a \in \mathcal{A} :\ f_a \text{ is of type 0 or of type I, } w_a \geq m_{2j-1} \text{ and} \\
&\qquad \text{for every } \beta \prec a \text{ with } f_\beta \text{ of type I, } w_\beta < m_{2j-1} \text{ and } w(f_\beta) \leq m_{2j-1}\}.
\end{aligned}$$

We set $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{B} = \{\beta \in \mathcal{A} :\ \text{there exists } \delta \in \mathcal{D} \text{ with } \beta \preceq \delta\}$. The following properties hold.

(i) The sets $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ are pairwise disjoint.
(ii) The nodes of $\mathcal{D}$ are pairwise incomparable with respect to the order of $\mathcal{A}$. Moreover, the set $\mathcal{D}$ is a maximal subset of $\mathcal{A}$ with this property. In order to see the later it is enough to show that every branch of the tree $\mathcal{A}$ contains a member of $\mathcal{D}$. Indeed, fix a branch of $\mathcal{A}$ and let $a_0 \prec a_1 \prec \cdots \prec a_k$ be the members of this branch for which the corresponding functionals are of type I or of type 0. If the set $\big\{i \in \{0, 1, \ldots, k\} :\ w_{a_i} \geq m_{2j-1} \text{ or } w(f_{a_i}) \geq m_{2j-1}\big\}$

is nonempty and i_0 is its minimum then $a_{i_0} \in \mathcal{D}$, while if this set is empty then $a_k \in \mathcal{D}$.

Thus $\mathcal{B}$ is a complete subtree of $\mathcal{A}$, while a node $\beta \in \mathcal{B}$ is maximal of $\mathcal{B}$ if and only if $\beta \in \mathcal{D}$.

(iii) For every $\delta \in \mathcal{D}$ the set $\{\beta \in \mathcal{B} :\ \beta \prec \delta$ and f_β is of type I$\}$ has cardinality at most $\log_2(m_{2j-1})$. Indeed, let $\{\beta_0 \prec \beta_1 \prec \cdots \prec \beta_{d-1}\}$ be the later set. Then $w_{\beta_{d-1}} < m_{2j-1}$ and since $w_{\beta_{d-1}} = w(f_{\beta_0}) \cdot \ldots \cdot w(f_{\beta_{d-2}}) \geq 2^{d-1}$, it follows that $d-1 < \log_2(m_{2j-1})$, therefore $d \leq \log_2(m_{2j-1})$.

(iv) For every $\beta \in \mathcal{B}$ with $\beta \notin \mathcal{D}$, such that the functional f_β is of type I, it holds that $w(f_\beta) < m_{2j-1}$. It follows from Corollary 3.9 that the node $\beta \in \mathcal{B}$ has at most $n_{2j-2}\log_2(m_{2j-1})$ immediate successors.

Using arguments similar to those in the proof of Lemma 4.5, we may prove the following. The functional f takes the form $f = \sum_{i=1}^{d} \lambda_i f_i$ as a convex combination, such that for each i there exists a family of functionals $(f^i_\beta)_{\beta\in\mathcal{B}^i}$, indexed by a finite tree $\mathcal{B}^i$ and an order preserving map $\Phi^i : \mathcal{B}^i \to \mathcal{B}$ with the following properties.

(v) The tree $\mathcal{B}^i$ has a unique root $0 \in \mathcal{B}^i$ and $f^i_0 = f_i$.

(vi) For every maximal node $\beta \in \mathcal{B}^i$, $\Phi^i(\beta)$ is a maximal node of $\mathcal{B}$ (and thus $\Phi^i(\beta) \in \mathcal{D}$) and $f^i_\beta = f_{\Phi^i(\beta)}$. In particular, $f^i_\beta \in D$.

(vii) For every $a \in \mathcal{B}^i$ which is non-maximal, the functionals f^i_a and $f_{\Phi^i(a)}$ are of type I with $w(f^i_a) = w(f_{\Phi^i(a)})$ and $\#S^i_a = \#S_{\Phi^i(a)}$, where S^i_a is the set of immediate successors of $a \in \mathcal{B}^i$ and S_γ denotes the set of immediate successors for a $\gamma \in \mathcal{B}$. Thus $f^i_a = \frac{1}{w(f^i_a)}(f^i_{\beta_1} + \cdots + f^i_{\beta_d})$ with $f^i_{\beta_1} < \cdots < f^i_{\beta_d}$ and $d < n_{2j-2}\log_2(m_{2j-1})$ (see property (iv) above). Moreover, defining $w^i_\beta = \prod\{w(f^i_a) :\ a \in \mathcal{B}^i,\ a \prec \beta\}$ for each $\beta \in \mathcal{B}^i$, we have that $w^i_\beta = w_{\Phi^i(\beta)}$.

(viii) Denoting by $\mathcal{B}^i_{\max}$ the set of maximal nodes of the tree $\mathcal{B}^i$, the functionals $(f^i_\beta)_{\beta\in\mathcal{B}^i_{\max}}$ are successive and $f = \sum_{\beta\in\mathcal{B}^i_{\max}} \frac{1}{w^i_\beta} f^i_\beta$.

We notice that for $a \in \mathcal{B}^i$ which is non-maximal, the functional f^i_β may not belong to the norming set D of the space $\mathfrak{X}_D$, but certainly belongs to the unconditional frame of the space, i.e. to the minimal subset of $c_{00}(\mathbb{N})$ which contains $\{\pm e^*_n :\ n \in \mathbb{N}\}$, is closed under the $(\mathcal{A}_{n_j}, \frac{1}{m_j})$ operations and is rationally convex.

From properties (iii), (iv), (vii) and the fact that the map Φ^i is order preserving we get the following.

(ix) The cardinality of the set $\mathcal{B}^i_{\max}$ of maximal nodes of $\mathcal{B}^i$ does not exceed the number $Q_{2j-1} = (n_{2j-2}\log_2(m_{2j-1}))^{\log_2(m_{2j-1})}$.

Since the functional f equals to the convex combination $\sum_{i=1}^{d} \lambda_i f_i$, in order to prove the lemma, it is enough to show that for each $i \in \{1, \ldots, d\}$, it holds that

$$|(f_i \cdot \phi_\chi)\big(\sum_{k\in I}(-1)^{k+1}x_k\big)| \leq \frac{4C}{m^2_{2j-1}} \cdot \#(I) + 2C \cdot Q_{2j-1}. \tag{4}$$

We fix $i \in \{1, \ldots, d\}$ and we set $E_\beta = \operatorname{ran}(f^i_\beta)$ for each $\beta \in \mathcal{B}^i_{\max}$. We partition the interval I as follows

$$I_1 = \{k \in I :\ E_\beta \cap \operatorname{ran}(x_k) \neq \emptyset \text{ for at most one } \beta \in \mathcal{B}^i_{\max}\}$$

$$I_2 = \{k \in I :\ E_\beta \cap \operatorname{ran}(x_k) \neq \emptyset \text{ for at least two } \beta \in \mathcal{B}^i_{\max}\}.$$

For each $\beta \in \mathcal{B}^i_{\max}$ we set $I_\beta = \{k \in I_1 : \ E_\beta \cap \mathrm{ran}(x_k) \neq \emptyset\}$. We observe the following.

(x) Each I_β is an interval and the intervals $(I_\beta)_{\beta\in\mathcal{B}^i_{\max}}$ are pairwise disjoint.

(xi) For each $\beta \in \mathcal{B}^i_{\max}$ we have that $E_\beta \cap \mathrm{ran}(x_k) \neq \emptyset$ for at most two $k \in I_2$.

For $p = 0, 1, 2$ we set $\mathcal{B}^{i,p}_{\max} = \{a \in \mathcal{B}^i_{\max} : \ \Phi^i(a) \in \mathcal{D}_p\}$. Observe that the sets $\mathcal{B}^{i,0}_{\max}, \mathcal{B}^{i,1}_{\max}, \mathcal{B}^{i,2}_{\max}$ form a partition of $\mathcal{B}^i_{\max}$. The proof of (4) (and the proof of the whole lemma) will be complete after showing the following.

$$|(f_i \cdot \phi_\chi)(\sum_{k\in I_2}(-1)^{k+1}x_k)| \quad \leq \quad C \cdot Q_{2j-1} \tag{5}$$

$$|((\sum_{\beta\in\mathcal{B}^{i,2}_{\max}} \frac{1}{w^i_\beta} f^i_\beta) \cdot \phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| \quad \leq \quad \frac{2C}{m^2_{2j-1}} \cdot \#(I) \tag{6}$$

$$|((\sum_{\beta\in\mathcal{B}^{i,1}_{\max}} \frac{1}{w^i_\beta} f^i_\beta) \cdot \phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| \quad \leq \quad \frac{2C}{m^2_{2j-1}} \cdot \#(I) \tag{7}$$

$$|((\sum_{\beta\in\mathcal{B}^{i,0}_{\max}} \frac{1}{w^i_\beta} f^i_\beta) \cdot \phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| \quad \leq \quad C \cdot Q_{2j-1}. \tag{8}$$

Let us first prove (5). From properties (viii), (ix), (xi), the fact that $\|x_k\| \leq 2C$ and since $f^i_\beta \cdot x^*_k \in D$ for each $\beta \in \mathcal{B}^i_{\max}$ and each k, we get that

$$\begin{aligned}
|(f_i \cdot \phi_\chi)(\sum_{k\in I_2}(-1)^{k+1}x_k)| &\leq \sum_{\beta\in\mathcal{B}^i_{\max}} \frac{1}{w^i_\beta}|(f^i_\beta \cdot \phi_\chi)(\sum_{k\in I_2}(-1)^{k+1}x_k)| \\
&\leq \frac{1}{m_{2j-1}} \sum_{\beta\in\mathcal{B}^i_{\max}} \sum_{k\in I_2} |(f^i_\beta \cdot x^*_k)(x_k)| \\
&\leq \frac{1}{m_{2j-1}} \cdot \#(\mathcal{B}^i_{\max}) \cdot 2 \cdot \max_k \|x_k\| \\
&\leq \frac{4C \cdot Q_{2j-1}}{m_{2j-1}} \leq C \cdot Q_{2j-1}.
\end{aligned}$$

We pass to the proof of (6). From (vii) and from the definitions of $\mathcal{B}^{i,2}_{\max}$, $\mathcal{D}_2$ we get that for $k \in \mathcal{B}^{i,2}_{\max}$ it holds that $w^i_\beta = w_{\Phi^i(\beta)} \geq m_{2j-1}$. From this and from the fact that the sets $(I_\beta)_{\beta\in\mathcal{B}^i_{\max}}$ are pairwise disjoint subsets of I, we get that

$$\begin{aligned}
|((\sum_{\beta\in\mathcal{B}^{i,2}_{\max}} \frac{1}{w^i_\beta} f^i_\beta) \cdot \phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| &\leq \sum_{\beta\in\mathcal{B}^{i,2}_{\max}} \frac{1}{w^i_\beta} \frac{1}{m_{2j-1}} \sum_{k\in I_1} |(f^i_\beta \cdot x^*_k)(x_k)| \\
&\leq \frac{1}{m^2_{2j-1}} \sum_{\beta\in\mathcal{B}^{i,2}_{\max}} \sum_{k\in I_\beta} \|x_k\| \\
&\leq \frac{1}{m^2_{2j-1}} \cdot \#(\bigcup_{\beta\in\mathcal{B}^{i,2}_{\max}} I_\beta) \cdot 2C \\
&\leq \frac{2C}{m^2_{2j-1}} \cdot \#(I).
\end{aligned}$$

Next we show (7). From the fact that each (x_k, x^*_k) is a $(C, 2j_k)$ exact pair, it follows that for every $\beta \in \mathcal{B}^{i,1}_{\max}$ we have that $|(f^i_\beta \cdot x^*_k)(x_k)| \leq C(\frac{m_{2j_k}}{w(f^i_\beta)w(x^*_k)} + \frac{1}{m_{2j_k}}) \leq$

$C(\frac{m_{2j_k}}{m_{2j-1}m_{2j_k}}+\frac{1}{m_{2j_k}})\leq\frac{2C}{m_{2j-1}}$ (we have used that $w(f^i_\beta)=w(f_{\Phi^i(\beta)})\geq m_{2j-1}$ as follows from (vii) and from the definitions of $\mathcal{B}^{i,1}_{\max}$, $\mathcal{D}_1$). This implies that

$$\begin{aligned}|((\sum_{\beta\in\mathcal{B}^{i,1}_{\max}}\frac{1}{w^i_\beta}f^i_\beta)\cdot\phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| &\leq \sum_{\beta\in\mathcal{B}^{i,1}_{\max}}\sum_{k\in I_\beta}\frac{1}{m_{2j-1}}|(f^i_\beta\cdot x^*_k)(x_k)|\\ &\leq \frac{1}{m_{2j-1}}\sum_{\beta\in\mathcal{B}^{i,1}_{\max}}\sum_{k\in I_\beta}\frac{2C}{m_{2j-1}}\\ &\leq \frac{2C}{m^2_{2j-1}}\cdot\#(I).\end{aligned}$$

Finally, we shall show (8). For each $\beta\in\mathcal{B}^{i,0}_{\max}$ the functional f^i_β is of the form $f^i_\beta=\pm\chi_{J_\beta}$ for some interval J_β. Taking into account that I_β is an interval and that $x^*_k(x_k)=1$ for each k, setting $k_1=\min I_\beta$, $k_2=\max I_\beta$ and $I'_\beta=I_\beta\setminus\{k_1,k_2\}$, we get that

$$\begin{aligned}|(f^i_\beta\cdot\phi_\chi)(\sum_{k\in I_\beta}(-1)^{k+1}x_k)| &\leq \frac{1}{m_{2j-1}}\big(\|x_{k_1}\|+\big|(\sum_{k\in I'_\beta}x^*_k)(\sum_{k\in I'_\beta}(-1)^{k+1}x_k)\big|\\ &\qquad +\|x_{k_2}\|\big)\\ &\leq \frac{1}{m_{2j-1}}\cdot(2C+1+2C)=\frac{4C+1}{m_{2j-1}}.\end{aligned}$$

Hence,

$$\begin{aligned}|((\sum_{\beta\in\mathcal{B}^{i,0}_{\max}}\frac{1}{w^i_\beta}f^i_\beta)\cdot\phi_\chi)(\sum_{k\in I_1}(-1)^{k+1}x_k)| &\leq \sum_{\beta\in\mathcal{B}^{i,0}_{\max}}\frac{1}{w^i_\beta}|(f^i_\beta\cdot\phi_\chi)(\sum_{k\in I_\beta}(-1)^{k+1}x_k)|\\ &\leq \frac{4C+1}{m_{2j-1}}\cdot\#(\mathcal{B}^{i,0}_{\max})\leq C\cdot Q_{2j-1}.\end{aligned}$$

From (5), (6), (7), (8) we get (4) and this completes the proof of the lemma. $\square$

With the next lemma we pass from the action of products of the form $f\cdot\phi_\chi$ for $f\in D$ on the vector $\sum_{k\in I}(-1)^{k+1}x_k$, to the action of $f\cdot\phi$ for $f\in D$ and arbitrary $\phi\in K$ with $w(\phi)=m_{2j-1}$ on the same vector.

Lemma 5.4. Let $f\in D$, let $\phi\in K^{2j-1}$ (which means that ϕ is of type I with $w(\phi)=m_{2j-1}$) and let $\chi=(x_k,x^*_k)_{k=1}^{n_{2j-1}}$ be a $(C,2j-1)$ dependent sequence. Then for every subinterval I of the interval $\{1,2,\dots,n_{2j-1}\}$, we have that

$$|(f\cdot\phi)\big(\sum_{k\in I}(-1)^{k+1}x_k\big)|\leq\frac{5C}{m^2_{2j-1}}\cdot\#(I)+3C\cdot Q_{2j-1}.$$

Proof. We may assume, without loss of generality, that the functional f is either of type I or of type 0 (since every member of D is s convex combination of such functionals). The functional ϕ takes the form

$$\phi=\frac{1}{m_{2j-1}}(Ex^*_t+x^*_{t+1}+\cdots+x^*_{r-1}+f_r+f_{r+1}+\cdots+f_d)$$

where $(x^*_1,x^*_2,\dots,x^*_{r-1},f_r,f_{r+1},\cdots,f_{n_{2j-1}})$ is some $2j-1$ special sequence, $w(f_r)=w(x^*_r)$, $f_r\neq x^*_r$, $d\leq n_{2j-1}$ and E is an interval of the form $[m,\max\operatorname{supp}x^*_t]$. We

set

$$\phi_1 = \frac{1}{m_{2j-1}}(Ex_t^* + x_{t+1}^* + \cdots + x_{r-1}^*) \quad \text{and} \quad \phi_2 = \frac{1}{m_{2j-1}}(f_r + f_{r+1} + \cdots + f_d).$$

We observe that $\phi_1 \cdot f = \big(\frac{1}{m_{2j-1}}(x_1^* + x_2^* + \cdots + x_{n_{2j-1}}^*)\big) \cdot \chi_{[\min E, \max\operatorname{supp} x_{r-1}^*]} \cdot f = \phi_\chi \cdot h$ where $h = \chi_{[\min E, \max\operatorname{supp} x_{r-1}^*]} \cdot f \in D$. From Lemma 5.3 it follows that

$$|(f\cdot\phi_1)\big(\sum_{k\in I}(-1)^{k+1}x_k\big)| \leq \frac{4C}{m_{2j-1}^2}\cdot \#(I) + 2C\cdot Q_{2j-1}. \tag{9}$$

We select p such that $w(x_{p-1}^*) < w(f) \leq w(x_p^*)$ (the adaptations in the rest of the proof are obvious if no such a p exists). From the injectivity of the coding function σ and the definition of special functionals (Definition 3.4) we get that the sets $\{w(f_{r+1}), \ldots, w(f_d)\}$ and $\{w(x_k^*) : k = 1, \ldots, n_{2j-1}\}$ are disjoint and both are subsets of the set $\{m_{2i} : i \in \mathbb{N}\}$.

Let $k \in I$, $k < p-1$. Then for every $i \in \{r, \ldots, d\}$ we have that $w(f\cdot f_i) \geq w(f) > w(x_{p-1}^*) \geq m_{2j_k}^5$, hence, using that (x_k, x_k^*) is a $(C, 2j_k)$ exact pair, we get that $|(f\cdot f_i)(x_k)| \leq C(\frac{m_{2j_k}}{w(f\cdot f_i)} + \frac{1}{m_{2j_k}}) \leq \frac{2C}{m_{2j_k}}$. Thus

$$|(f\cdot\phi_2)(x_k)| \leq \frac{1}{m_{2j-1}}\sum_{i=r}^{d}|(f\cdot f_i)(x_k)| \leq \frac{1}{m_{2j-1}}\cdot n_{2j-1}\cdot\frac{2C}{m_{2j_k}} \leq \frac{C}{m_{2j-1}^2}.$$

Let now $k \in I$, $k > p$, $k \neq r$. We observe that for i such that $w(f_i) < m_{2j_k}$ we also have that $w(f\cdot f_i) = w(f)\cdot w(f_i) \leq m_{2j_p}\cdot m_{2j_k-1} < m_{2j_k}$. Thus setting $J_-^k = \{i : r \leq i \leq d, \ w(f_i) < m_{2j_k}\}$, $J_+^k = \{i : r \leq i \leq d, \ w(f_i) > m_{2j_k}\}$ and taking into account that (x_k, x_k^*) is a $(C, 2j_k)$ exact pair, we get that

$$\begin{aligned}
|(f\cdot\phi_2)(x_k)| &\leq \frac{1}{m_{2j-1}}\big(\sum_{i\in J_-^k}|(f\cdot f_i)(x_k)| + \sum_{i\in J_+^k}|(f\cdot f_i)(x_k)|\big) \\
&\leq \frac{1}{m_{2j-1}}\big(\sum_{i\in J_-^k}\frac{3C}{w(f)\cdot w(f_i)} + \sum_{i\in J_+^k}C(\frac{m_{2j_k}}{w(f)\cdot w(f_i)} + \frac{1}{m_{2j_k}})\big) \\
&\leq \frac{C}{m_{2j-1}}\big(\sum_{i\in J_-^k}\frac{3}{w(f_i)} + m_{2j_k}\sum_{i\in J_+^k}\frac{1}{w(f_i)} + n_{2j-1}\cdot\frac{1}{m_{2j_k}}\big) \\
&\leq \frac{C}{m_{2j-1}^2}.
\end{aligned}$$

Thus, setting $I_1 = I \cap \{p-1, p, r\}$ and $I_2 = I \setminus \{p-1, p, r\}$ we get that

$$|(f\cdot\phi_2)\big(\sum_{k\in I}(-1)^{k+1}x_k\big)| \leq \sum_{k\in I_1}\|x_k\| + \sum_{k\in I_2}\frac{C}{m_{2j-1}^2} \leq 6C + \frac{C}{m_{2j-1}^2}\cdot\#(I). \tag{10}$$

From (9), (10) we conclude that

$$|(f\cdot\phi)\big(\sum_{k\in I}(-1)^{k+1}x_k\big)| \leq \frac{5C}{m_{2j-1}^2}\cdot\#(I) + 3C\cdot Q_{2j-1}.$$

$\square$

Proposition 5.5. If $(x_k, x_k^*)_{k=1}^{n_{2j-1}}$ is a $(C, 2j-1)$ dependent sequence, then

$$\|\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k\| \leq \frac{24C}{m_{2j-1}^2}.$$

Proof. From Remark 5.2 we get that the block sequence $(x_k)_{k=1}^{n_{2j-1}}$ is a $(3C, \frac{1}{m_{2j-1}^2})$ RIS, hence the same holds for the sequence $\big((-1)^{k+1}x_k\big)_{k=1}^{n_{2j-1}}$. From Lemma 5.4 we have that for every $h \in K^{2j-1} \cdot D$ and every subinterval I of the interval $\{1, 2, \dots, n_{2j-1}\}$ it holds that

$$|h\big(\sum_{k \in I} (-1)^{k+1} x_k\big)| \leq \frac{5C}{m_{2j-1}^2} \cdot \#(I) + 3C \cdot Q_{2j-1}.$$

Hence for h and I as above, with $\#(I) \geq m_{2j-1}^2 Q_{2j-1}$, we have that

$$|h\big(\sum_{k \in I} (-1)^{k+1} x_k\big)| \leq 8C \cdot \frac{1}{m_{2j-1}^2} \cdot \#(I).$$

Thus the basic inequality (Proposition 4.8) with the additional assumption is applicable to the sequence $\big((-1)^{k+1}x_k\big)_{k=1}^{n_{2j-1}}$.

Let $f \in D$. Then there exists $g \in W$, satisfying

$$|f(\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k)| \leq 8C\big(g(\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} e_k) + \frac{1}{m_{2j-1}^2}\big)$$

and such that the functional g admits a tree $(g_a)_{a \in \mathcal{A}}$ that for every $a \in \mathcal{A}$ with g_a of type I and $w(g_a) < m_{2j-1}^2$, the node a has at most $m_{2j-1}^2 Q_{2j-1}$ immediate successors. From Lemma 4.7, it follows that

$$|g(\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} e_k)| \leq \frac{2}{m_{2j-1}^2}$$

therefore

$$|f(\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k)| \leq \frac{24C}{m_{2j-1}^2}.$$

Since this happens for every $f \in D$ the conclussion follows. $\square$

Theorem 5.6. The space $\mathfrak{X}_D$ is Hereditarily Indecomposable.

Proof. Let Y, Z be a pair of block subspaces of $\mathfrak{X}_D$, and let $\delta > 0$. We choose $j \in \mathbb{N}$ with $m_{2j-1} > \frac{96}{\delta}$.

Using Lemma 4.16 we may inductively select a $(4, 2j-1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{2j-1}}$ (Definition 5.1) such that $x_{2k-1} \in Y$ and $x_{2k} \in Z$ for $1 \leq k \leq \frac{n_{2j-1}}{2}$. We set

$$y = \frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}/2} x_{2k-1} \in Y \quad \text{and} \quad z = \frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}/2} x_{2k} \in Z.$$

Since the functional $\phi_\chi = \frac{1}{m_{2j-1}} \sum\limits_{k=1}^{n_{2j-1}} x_k^*$ satisfies $\|\phi_\chi\| \leq 1$ we get that $\|y+z\| \geq \phi_\chi(y+z) = \frac{1}{m_{2j-1}}$. On the other hand Proposition 5.5 implies that $\|y-z\| \leq \frac{96}{m_{2j-1}^2}$.

Therefore $\|y-z\| \leq \delta \cdot \|y+z\|$ and this finishes the proof of the theorem. $\square$

6. The Banach algebras $\mathfrak{X}_D^*$, $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ are HI

In this section, we initially prove that the space $(\mathfrak{X}_D)_* = \overline{\text{span}}\{e_n^* : n \in \mathbb{N}\}$ is HI. This, in conjunction to the fact that $\dim(\mathfrak{X}_D^*/(\mathfrak{X}_D)_*) = 1$, entails that $\mathfrak{X}_D^*$ is also HI. From Proposition 3.13 we know that the Banach algebras $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ and $\mathfrak{X}_D^*$ are isometric, hence $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is also HI. We also notice (as follows from Remark 2.6) that the algebra $\mathcal{K}_{\text{diag}}(\mathfrak{X}_D)$, i.e. the algebra of compact diagonal operators on the space $\mathfrak{X}_D$, is isometric to the space $(\mathfrak{X}_D)_*$.

Definition 6.1. Let $C > 1$ and $k \in \mathbb{N}$. A finitely supported vector $x^* \in (\mathfrak{X}_D)_*$ is said to be a $C - c_0^k$ vector if $\|x^*\| \leq 1$ and x^* takes the form $x^* = x_1^* + x_2^* + \cdots + x_k^*$ with $x_1^* < x_2^* < \cdots < x_k^*$ and $\|x_i^*\| \geq C^{-1}$.

Lemma 6.2. Let Z be a block subspace of $(\mathfrak{X}_D)_*$ and let $N \in \mathbb{N}$. Then there exists a block sequence $(x_n^*)_{n\in\mathbb{N}}$ in Z with $\|x_n^*\| \geq 1$ such that for every $I \in \mathbb{N}^{[N]}$ and every choice of signs $(\varepsilon_i)_{i\in I} \in \{-1,1\}^I$ we have that $\|\sum\limits_{n\in I} \varepsilon_n x_n\| < 2$.

(For an infinite set L we denote by $L^{[N]}$ the set of all subsets of L having N elements and by $[L]$ the set of all infinite subsets of L.)

Proof. Assume that the lemma fails. We select s, j with $2^s > m_{2j}$ and $N^s \leq n_{2j}$. We shall denote by $\mathbb{N}_0$ the set $\{0, 1, 2, \ldots\}$. We choose an arbitrary normalized block sequence $(f_i^0)_{i\in\mathbb{N}_0}$ in the block subspace Z.

We set

$$\mathcal{A}_1 = \{L \in [\mathbb{N}_0],\ L = \{l_i : i \in \mathbb{N}_0\} : \forall (\varepsilon_i)_{i=0}^{N-1} \in \{-1,1\}^N,\ \ \|\sum_{i=0}^{N-1} \varepsilon_i f_{l_i}^0\| < 2\}$$

$$\begin{aligned}
\mathcal{B}_1 &= [\mathbb{N}_0] \setminus \mathcal{A}_1 \\
&= \{L \in [\mathbb{N}_0],\ L = \{l_i : i \in \mathbb{N}_0\} : \exists (\varepsilon_i)_{i=0}^{N-1} \in \{-1,1\}^N,\ \ \|\sum_{i=0}^{N-1} \varepsilon_i f_{l_i}^0\| \geq 2\}.
\end{aligned}$$

From Ramsey' s theorem, there exists a homogenous set L either in $\mathcal{A}_1$ or in $\mathcal{B}_1$. Our assumption on the failure of the lemma rejects the first alternative, hence the homogenous set is in $\mathcal{B}_1$. We may assume that $L = \mathbb{N}_0$. In particular we get that there exist $\varepsilon_i^0 \in \{-1,1\}$, $i \in \mathbb{N}_0$, such that setting $f_n^1 = \sum\limits_{i=nN}^{(n+1)N-1} \varepsilon_i^0 f_i^0$, $n \in N_0$, we have that $\|f_n^1\| \geq 2$ for all n.

We set

$$\mathcal{A}_2 = \{L \in [\mathbb{N}_0],\ L = \{l_i : i \in \mathbb{N}_0\} : \forall (\varepsilon_i)_{i=0}^{N-1} \in \{-1,1\}^N,\ \ \|\sum_{i=0}^{N-1} \varepsilon_i f_{l_i}^1\| < 2^2\}$$

$$\begin{aligned}
\mathcal{B}_2 &= [\mathbb{N}_0] \setminus \mathcal{A}_2 \\
&= \{L \in [\mathbb{N}_0],\ L = \{l_i : i \in \mathbb{N}_0\} : \exists (\varepsilon_i)_{i=0}^{N-1} \in \{-1,1\}^N,\ \ \|\sum_{i=0}^{N-1} \varepsilon_i f_{l_i}^1\| \geq 2^2\}.
\end{aligned}$$

Again, the homogenous set L resulting from Ramsey' s theorem can not be in $\mathcal{A}_2$, since then the sequence $(\frac{1}{2} f_n^1)_{n\in L}$ would satisfy the conclusion of the lemma and this contradicts to our assumption that the lemma fails. As before, we may assume

that $L=\mathbb{N}_0$; we choose $\varepsilon_i^1\in\{-1,1\}$, $i\in\mathbb{N}_0$, such that the functionals $f_n^2=\sum_{i=nN}^{(n+1)N-1}\varepsilon_i^1 f_i^1$, $n\in N_0$ satisfy $\|f_n^2\|\geq 2^2$. Notice that $f_n^2=\sum_{i=nN^2}^{(n+1)N^2-1}\varepsilon_i^0\varepsilon^1_{[\frac{i}{N}]}f_i^0$.

After s consecutive applications of the same argument we obtain a block sequence $(f_n^s)_{n\in N_0}$ with $\|f_n^s\|\geq 2^s$ such that $f_n^s=\sum_{i=nN^s}^{(n+1)N^s-1}\delta_i f_i^0$ for some sequence of signs $(\delta_i)_{i\in\mathbb{N}_0}$. Taking into account that $N^s\leq n_{2j}$, Remark 3.5 implies that $\|\frac{1}{m_{2j}}\sum_{i=nN^s}^{(n+1)N^s-1}\delta_i f_i^0\|\leq 1$, i.e. $\|f_n^s\|\leq m_{2j}$. We thus get that $2^s\leq\|f_n^s\|\leq m_{2j}$ which contradicts to our choice of s,j. The proof of the lemma is complete. □

Lemma 6.3. Let Z be a block subspace of $(\mathfrak{X}_D)_*$, $\varepsilon>0$ and $k\in\mathbb{N}$. Then there exist z^* a $2-c_0^k$ vector with $z^*\in Z$ and z a $2-\ell_1^k$ average such that $\operatorname{ran} z^*=\operatorname{ran} z$, $z^*(z)>1$ and $\|z\|_G<\varepsilon$.

Proof. We choose d with $\frac{9}{2}\cdot\frac{1}{d}<\varepsilon$ and we set $N=k\cdot(2d)$. Applying Lemma 6.2 we select a block sequence $(x_n^*)_{n\in\mathbb{N}}$ in Z, with $\|x_n^*\|>\frac{1}{2}$, such that for every subset I of $\mathbb{N}$ with N elements and every choice of signs $(\varepsilon_n)_{n\in I}\in\{-1,1\}^I$ we have that $\|\sum_{n\in I}\varepsilon_n x_n^*\|\leq 1$. For each n, we select $x_n\in\mathfrak{X}_D$ with $\operatorname{ran} x_n=\operatorname{ran} x_n^*$, $\|x_n\|\leq 1$ and $x_n^*(x_n)>\frac{1}{2}$. We notice that for every subset I of $\mathbb{N}$ with N elements and every choice of scalars $(\lambda_n)_{n\in I}$ we have that $\|\sum_{n\in I}\lambda_n x_n\|\geq\frac{1}{2}\sum_{n\in I}|\lambda_n|$, due to the action of the functional $\sum_{n\in I}\varepsilon_n x_n^*$, where $\varepsilon_n=\operatorname{sgn}(\lambda_n)$.

We may assume, passing to a subsequence, that the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly Cauchy in Y_G, hence the sequence of its successive differences, i.e. the sequence $(y_n)_{n\in\mathbb{N}}$ defined as $y_n=x_{2n-1}-x_{2n}$, is weakly null in Y_G. We notice that the extreme points of the unit ball of the dual space $\mathrm{B}_{Y_G^*}$ are contained in the set $\overline{G}^p=\{\pm\chi_E:\ E\text{ is an interval of }\mathbb{N}\}$. Thus in order to check the behavior of a block sequence in the weak topology in Y_G, it is enough to check the action of $\pm\chi_{\mathbb{N}}$. Passing to a further subsequence we may assume that $\sum_{n=1}^{\infty}|\chi_{\mathbb{N}}(y_n)|<\frac{1}{2}$.

We claim that $\|\sum_{i=1}^m\varepsilon_i y_{k_i}\|_G\leq\frac{9}{2}$ for every $m\in\mathbb{N}$, $k_1<\cdots<k_m$ in $\mathbb{N}$ and every choice of signs $\varepsilon_1,\ldots,\varepsilon_m\in\{-1,1\}$. Indeed, let E be any finite interval. We denote by r (resp. s) the minimum (resp. maximum) integer i such that $E\cap\operatorname{ran} y_{k_i}\neq\emptyset$. Then for $r<i<s$ we have that $\chi_E(y_{k_i})=\chi_{\mathbb{N}}(y_{k_i})$ hence

$$\begin{aligned}
|\chi_E(\sum_{i=1}^m\varepsilon_i y_{k_i})| &\leq \|Ey_r\|_G+|\chi_{\mathbb{N}}(\sum_{i=r+1}^{s-1}\varepsilon_i y_{k_i})|\|+\|Ey_s\|_G\\
&\leq \|x_{2r-1}\|_G+\|x_{2r}\|_G+\sum_{i=r+1}^{s-1}|\chi_{\mathbb{N}}(y_{k_i})|+\|x_{2s-1}\|_G+\|x_{2s}\|_G\\
&\leq \|x_{2r-1}\|+\|x_{2r}\|+\sum_{n=1}^{\infty}|\chi_{\mathbb{N}}(y_n)|+\|x_{2s-1}\|+\|x_{2s}\|\\
&< 1+1+\frac{1}{2}+1+1=\frac{9}{2}
\end{aligned}$$

(with the obvious adaptations in the previous proof if $r=s-1$ or $r=s$).

For $i = 1, \dots, k$ we set

$$z_i^* = \sum_{l=(i-1)d+1}^{id} (x_{2l-1}^* - x_{2l}^*) \quad \text{and} \quad z_i = \frac{1}{d} \sum_{l=(i-1)d+1}^{id} y_l = \frac{1}{d} \sum_{l=(i-1)d+1}^{id} (x_{2l-1} - x_{2l}).$$

For each i we have that $\|z_i^*\| \geq \| - x_{2id}^*\| > \frac{1}{2}$ (due to the bimonotonicity of the norm), $\|z_i\| \leq \frac{1}{d} \sum_{l=(i-1)d+1}^{id} (\|x_{2l-1}\| + \|x_{2l}\|) \leq 2$, while

$$\|z_i\|_G \leq \frac{1}{d} \| \sum_{l=(i-1)d+1}^{id} (x_{2l-1} - x_{2l}) \|_G \leq \frac{1}{d} \cdot \frac{9}{2} < \varepsilon.$$

We also have that

$$z_i^*(z_i) = \frac{1}{d} \sum_{l=(i-1)d+1}^{id} \big(x_{2l-1}^*(x_{2l-1}) + x_{2l}^*(x_{2l}) \big) > \frac{1}{d} \sum_{l=(i-1)d+1}^{id} (\frac{1}{2} + \frac{1}{2}) = 1$$

and $\operatorname{ran} z_i^* = \operatorname{ran} z_i$.

Finally, we set

$$z^* = \sum_{i=1}^{k} z_i^* \quad \text{and} \quad z = \frac{1}{k} \sum_{i=1}^{k} z_i.$$

The fact that the functional z^* is the sum of $k \cdot (2d) = N$ functionals $\pm x_n^*$ and our initial choice of the sequence sequence $(x_n^*)_{n \in \mathbb{N}}$, imply that $\|z^*\| \leq 1$ while, since $\|z_i^*\| \geq \frac{1}{2}$ for each i, we get that z^* is a $2 - c_0^k$ vector belonging to the block subspace Z. We also have that $z^*(z) = \frac{1}{k} \sum_{i=1}^{k} z_i^*(z_i) > 1$ and $\operatorname{ran} z^* = \operatorname{ran} z$. Since $\|z\| \geq z^*(z) > 1$ and $\|z_i\| \leq 2$ for $i = 1, \dots, k$ the vector z is a $2 - \ell_1^k$ average, with $\|z\|_G \leq \frac{1}{k} \sum_{i=1}^{k} \|z_i\|_G < \varepsilon$. □

Corollary 6.4. Let Z be a block subspace of $(\mathfrak{X}_D)_*$, $k \in \mathbb{N}$ and $\varepsilon, \delta > 0$. Then there exist z a $2 - \ell_1^k$ average with $\|z\|_G < \varepsilon$ and $f \in D$ with $\operatorname{dist}(f, Z) < \delta$, such that $\operatorname{ran} f = \operatorname{ran} z$ and $f(z) > 1$.

Proof. Let z and z^* be the $2 - \ell_1^k$ average and the $2 - c_0^k$ vector respectively resulting from Lemma 6.3. Since the norming set D is pointwise dense in the unit ball of the dual space, we may choose $f \in D$ with $\operatorname{ran} f = \operatorname{ran} z^*$ such that $\|f - z^*\| < \min\{\delta, \frac{z^*(z)-1}{2}\}$. It is easy to check that z and f satisfy the conclusion of the corollary. □

Lemma 6.5. Let Z be a block subspace of $(\mathfrak{X}_D)_*$, $j \in N$ and $\delta > 0$. Then there exists a $(4, 2j)$ exact pair (z, z^*), with $\operatorname{dist}(z^*, Z) < \delta$.

Proof. Using Corollary 6.4, we may choose a sequence $(z_k, z_k^*)_{k \in \mathbb{N}}$ such that:

(i) The sequence $(z_k)_{k \in \mathbb{N}}$ is a block sequence in $\mathfrak{X}_D$ with $\|z_k\|_G < \frac{1}{m_{2j}^3}$ and each z_k is a $2 - \ell_1^{n_{j_k}}$ for an increasing sequence $(j_k)_k$.

(ii) $z_k^* \in D$ with $\operatorname{dist}(z_k^*, Z) < \frac{\delta}{n_{2j}}$.

(iii) $\operatorname{ran} z_k^* = \operatorname{ran} z_k$ and $z_k^*(z_k) > 1$.

From Remark 4.13 we may assume, passing to a subsequence, that $(z_k)_{k=1}^{n_{2j}}$ is a $(4, \frac{1}{m_{2j}^3})$ RIS. We set

$$z^* = \frac{1}{m_{2j}}(z_1^* + z_2^* + \cdots + z_{n_{2j}}^*).$$

Then the functional $z^* \in D$ is of type I, with $w(z^*) = m_{2j}$ and $\mathrm{dist}(z^*, Z) \leq \frac{1}{m_{2j}} \sum_{k=1}^{n_{2j}} \mathrm{dist}(z_k^*, Z) < \delta$.

From Corollary 4.9, for $f \in D$ of type I, we have that

$$|f\big(\frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k\big)| \leq \begin{cases} \frac{3\cdot 4}{w(f) m_{2j}} & \text{if } w(f) < m_{2j} \\ 4(\frac{1}{w(f)} + \frac{1}{m_{2j}^2}) & \text{if } w(f) \geq m_{2j}. \end{cases}$$

In particular $\|\frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k\| \leq \frac{8}{m_{2j}}$. On the other hand

$$z^*\big(\frac{1}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k\big) = \frac{1}{m_{2j}} \cdot \frac{1}{n_{2j}} \cdot \sum_{k=1}^{n_{2j}} z_k^*(z_k) > \frac{1}{m_{2j}}.$$

Thus there exists θ, with $\frac{1}{8} \leq \theta < 1$, such that $z^*\big(\theta \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} z_k\big) = 1$. We set

$$z = \theta \frac{m_{2j}}{n_{2j}}(z_1 + z_2 + \cdots + z_{n_{2j}}).$$

Then $z^*(z) = 1$, $\|z\|_G \leq \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} \|z_k\|_G < \frac{1}{m_{2j}^2}$, while for $f \in D$ of type I we have that

$$|f(z)| \leq \begin{cases} \frac{3\cdot 4}{w(f)} & \text{if } w(f) < m_{2j} \\ 4(\frac{m_{2j}}{w(f)} + \frac{1}{m_{2j}}) & \text{if } w(f) \geq m_{2j}. \end{cases}$$

Therefore (z, z^*) is a $(4, 2j)$ exact pair (Definition 4.15) with $\mathrm{dist}(z^*, Z) < \delta$. □

Theorem 6.6. The Banach space $(\mathfrak{X}_D)_*$ is Hereditarily Indecomposable.

Proof. Let Y, Z be a pair of block subspaces of $(\mathfrak{X}_D)_*$ an let $j \in \mathbb{N}$. Using Lemma 6.5 we may find a $(4, 2j-1)$ dependent sequence $(x_k, x_k^*)_{k=1}^{n_{2j-1}}$ (see Definition 5.1) which satisfies $\sum_k \mathrm{dist}(x_{2k-1}^*, Y) < 1$ and $\sum_k \mathrm{dist}(x_{2k}^*, Z) < 1$.

We set

$$y^* = \frac{1}{m_{2j-1}} \sum_{k=1}^{n_{2j-1}/2} x_{2k-1}^* \quad \text{and} \quad z^* = \frac{1}{m_{2j-1}} \sum_{k=1}^{n_{2j-1}/2} x_{2k}^*.$$

The functional $\phi_\chi = \frac{1}{m_{2j-1}} \sum_{k=1}^{n_{2j-1}} x_k^*$ satisfies $\|\phi_\chi\| \leq 1$ i.e. $\|y^* + z^*\| \leq 1$. Proposition 5.5 entails that $\|\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k\| \leq \frac{96}{m_{2j-1}^2}$ while $(y^* - z^*)(\frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k) = \frac{1}{m_{2j-1}}$, therefore $\|y^* - z^*\| \geq \frac{m_{2j-1}}{96}$.

Selecting $f_Y \in Y$ with $\|f_Y - y^*\| < 1$ and $f_Z \in Z$ with $\|f_Z - z^*\| < 1$, we get that $\|f_Y + f_Z\| < 3$ and $\|f_Y - f_Z\| > \frac{m_{2j-1}}{96} - 2$. Since this procedure may

be done for arbitrary large j, we conclude that the space $(\mathfrak{X}_D)_*$ is Hereditarily Indecomposable. $\square$

Theorem 6.7. The Banach algebras $\mathfrak{X}_D^*$ and $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ are Hereditarily Indecomposable.

Proof. From Proposition 4.18 the quotient space $\mathfrak{X}_D^*/(\mathfrak{X}_D)_*$ has dimension equal to one. Thus the fact that $(\mathfrak{X}_D)_*$ is Hereditarily Indecomposable (Theorem 6.6) implies that $(\mathfrak{X}_D)^*$ is also Hereditarily Indecomposable (see also Theorem 1.4 of [8]). As $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is isometric to $\mathfrak{X}_D^*$ we conclude that the Banach algebra $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ is also Hereditarily Indecomposable. $\square$

Remark 6.8. Let $\mathfrak{X}_{D,r}$ be the Banach space defined similarly to the space $\mathfrak{X}_D$, with the only difference concerning the first inductive step of the definition of its norming set, replacing the set $G = \{\pm\chi_I : \ I \text{ is a finite interval of } \mathbb{N}\}$ with the set $G_0 = \{\pm e_k^* : \ k \in \mathbb{N}\}$. Then the space $\mathfrak{X}_{D,r}$ is reflexive and HI while $\mathfrak{X}_{D,r}^*$ is an example of a reflexive HI Banach algebra. The reason we have included the set G in the norming set D of the space $\mathfrak{X}_D$, is in order to apply Theorem 2.4 and to obtain a HI Banach algebra of diagonal operators.

Theorem 6.9. Every diagonal operator $T : \mathfrak{X}_D \to \mathfrak{X}_D$ is of the form $T = \lambda I + K$ with the operator K being compact.

Proof. From Remark 2.6, the isometry $\Phi : \mathfrak{X}_D^* \to \mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$ of Theorem 2.4, carries the predual space $(\mathfrak{X}_D)_*$ onto the space $\mathcal{K}_{\text{diag}}(\mathfrak{X}_D)$ of compact diagonal operators of the space $\mathfrak{X}_D$. But since $(\mathfrak{X}_D)^* = (\mathfrak{X}_D)_* \oplus \text{span}\{\chi_{\mathbb{N}}\}$ and $\Phi(\chi_{\mathbb{N}}) = I$ we get that $\mathcal{L}_{\text{diag}}(\mathfrak{X}_D) = \mathcal{K}_{\text{diag}}(\mathfrak{X}_D) \oplus \text{span}\{I\}$ hence every diagonal operator $T : \mathfrak{X}_D \to \mathfrak{X}_D$ takes the form $T = \lambda I + K$ with K being a compact operator. $\square$

Remark 6.10. Since $\mathfrak{X}_D$ is the dual of the space $(\mathfrak{X}_D)_* = \overline{\text{span}}\{e_n^* : \ n \in \mathbb{N}\}$, observing that every $T \in \mathcal{L}_{\text{diag}}(\mathfrak{X}_D, (e_n)_{n\in\mathbb{N}})$, being $w^* - w^*$ continuous, is a dual operator, we get the following. The correspondence

$$\mathcal{L}_{\text{diag}}((\mathfrak{X}_D)_*) \ \ni \ R \quad \longrightarrow \quad R^* \ \in \ \mathcal{L}_{\text{diag}}(\mathfrak{X}_D)$$

is an onto isometry, while, restricting this correspondence to the subalgebras of compact diagonal operators, we obtain that $\mathcal{K}_{\text{diag}}((\mathfrak{X}_D)_*)$ is isometric to $\mathcal{K}_{\text{diag}}(\mathfrak{X}_D)$, which in turn is isometric to $(\mathfrak{X}_D)_*$ (Remark 2.6). Thus we have established the existence of a Banach space Y with a Schauder basis (namely $Y = (\mathfrak{X}_D)_*$ with the basis $(e_n^*)_{n\in\mathbb{N}}$) which is naturally isometric to the space $\mathcal{K}_{\text{diag}}(Y)$ of its compact diagonal operators, with the last being of codimension 1 in $\mathcal{L}_{\text{diag}}(Y)$. As we have noticed in the introduction, since the basis of Y is shrinking, the space $\mathcal{L}_{\text{diag}}(Y)$ is naturally identified with the second dual of $\mathcal{K}_{\text{diag}}(Y)$ ([18]).

References

[1] G. Androulakis, Th. Schlumprecht, *Strictly singular non-compact operators exist on the space of Gowers and Maurey*, J. London Math. Soc. (2) **64**, (2001), no. 3, 655–674.

[2] S.A. Argyros, A. Arvanitakis, A. Tolias, *Saturated extensions, the attractors method and Hereditarily James tree spaces*, Methods in Banach space Theory (edited by J.M.F Castillo and W.B. Johnson), 1–90, London Math. Soc. Lecture Note Ser. **337**, (2006).

[3] S.A. Argyros, I. Deliyanni, A. Tolias, *Strictly singular non-compact diagonal operators on HI spaces*, (preprint, arXiv:0807.2388).

[4] S.A. Argyros, R. G. Haydon, *A Hereditarily Indecomposable $\mathcal{L}^\infty$ space that solves the Scalar-plus-Compact problem* (preprint).

[5] S.A. Argyros, J. Lopez-Abad, S. Todorcevic, *A class of Banach spaces with few non-strictly singular operators*, J. Funct. Anal. **222**, (2005), no. 2, 306–384.
[6] S.A. Argyros, A. Manoussakis, *An Indecomposable and Unconditionally Saturated Banach space*, Studia Math., **159**, (2003), no. 1, 1–32.
[7] S.A. Argyros, S. Todorcevic, *Ramsey Methods in Analysis*, *Advance Cources in Mathematics CRM Barcelona*, Birkhauser, (2004).
[8] S.A. Argyros, A. Tolias, *Methods in the Theory of Hereditarily Indecomposable Banach Spaces*, Memoirs of the AMS, **170**, (2004), no. 806, vi+114pp.
[9] S.A. Argyros, A. Tolias, *Indecomposability and Unconditionality in duality*, *Geom. and Funct. Anal.*, **14**, (2004), no. 2, 247-282.
[10] K. Beanland, *Operators on asymptotic ℓ_p spaces which are not compact perturbations of a multiple of the identity*, Illinois J. Math., (to appear)
[11] S. Bellenot, R. Haydon, E. Odell, *Quasi-reflexive and tree spaces constructed in the spirit of R. C. James*, Banach space theory, *Contemp. Math. AMS*, **85**, (1989), 19-43.
[12] I. Gasparis, *Strictly singular non-compact operators on Hereditarily Indecomposable Banach spaces* Proc. of the A.M.S., **131**, (2003), no. 4, 1181-1189.
[13] W.T. Gowers, *A remark about the scalar-plus-compact problem, Proc. of the conference on Convex Geometric Analysis*, (Berkerey, CA, 1996), 111–115.
[14] W.T. Gowers, B. Maurey, *The Unconditional basic Sequence Problem*, *Journal of A.M.S.*, **6**, (1993), no. 4, 851–874.
[15] R.C. James, *A separable somewhat reflexive space with nonseparable dual*, *Bull. Amer. Math. Soc.*, **80**, (1974), 738–743.
[16] J. Lindenstrauss, C. Stegall, *Examples of separable spaces which do not contain ℓ_1 and whose duals are non-separable*, *Studia Math.*, **54**, (1975), no. 1, 81–105.
[17] E. Odell, H. P. Rosenthal, *A double-dual characterization of separable Banach spaces containing ℓ_1*, *Israel J. Math.*, **20**, (1975), no. 3-4, 375–384.
[18] A. Sersouri, *Opérateurs diagonaux dans les espaces á bases*, *Math. Z.* **199**, (1988), no. 4, 491–499.

(S.A. Argyros) Department of Mathematics, National Technical University of Athens
E-mail address: sargyros@math.ntua.gr

(I. Deliyanni) 18 Neapoleos St., Ag. Paraskevi, Athens
E-mail address: ideliyanni@yahoo.gr

(A. Tolias) Department of Mathematics, University of the Aegean
E-mail address: atolias@math.aegean.gr